



# A STUDY OF $h$ - PURE SUBMODULES

**ABSTRACT**

**THESIS**

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IN

**MATHEMATICS**

BY

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UNDER THE SUPERVISION OF

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ALIGARH (INDIA)  
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## Abstract

It is well known that pure subgroups, neat subgroups, basic subgroups, high subgroups, large subgroups and divisible groups etc. have become most useful tools in abelian groups. Most of these concepts have been generalized for modules by different Mathematicians. These studies were carried by imposing some restrictions either on modules or the ring involved. More often, the latter type of study has been made. For instance, different mathematician, Kaplansky, R. Baer, E. Matlis, J.C. Robson, H. Marubayashi etc., studied modules over PID, discrete valuation ring, Dedekind prime ring, generalized uniserial rings etc. This work has been mainly influenced by outstanding researches done by P. Hill, C. Megibben, J.M. Irwin, E.A. Walker, Khalid Ben Abdullah, K. Honda, J.D. Moore, S. Singh and M.Z. Khan.

S. Singh [44] studied  $h$ -pure submodule of a unital module with two conditions and generalized some results of abelian groups. Since then a number of papers have been written in connection with the generalizations of the results of abelian groups. For instance, Khalid Ben Abdullah [6,7], M.H. Upham [47], S. Singh [44,45,46], M. Zubair Khan [15,16,17,18,20,22] generalized the fundamental concepts and results of abelian groups. These modules were called as  $S_2$ -modules [22] or TAG-module [6]. In [45] S. Singh studied QTAG-module and got some nice results.

The main aim of the present thesis is to study the  $h$ -pure submodule of a QTAG-module. We have also introduced some new concepts and obtained different characterizations.

The present thesis comprises four chapters having various results. The first section of each chapter provides an introduction to its contents. The number like 2.2.4 indicates result 4 of section 2 of chapter 2. The numbers in bracket refer to the references listed in the bibliography. Many result of this thesis have been published or accepted for the publications.

Chapter 1 deals with some basic and elementary notions and results. A brief survey of some of the known results is also included. This is mainly done to fix up the terminology and some other background informations needed for subsequent chapters.

In Chapter 2, we have studied about the quasi-essential submodules. We proved that a submodule  $N$  of a QTAG-module  $M$  is quasi-essential if and only if  $K/T$  is an absolute summand of  $M/T$  where  $K$  is an  $h$ -pure submodule of  $M$  containing  $N$  and  $T$  is a complement of  $K$  (Proposition 2.2.10). Further we established various conditions under which  $h$ -pure submodules are direct summands. We have further introduced the concept of essentially finitely indecomposable QTAG-module and proved that every  $h$ -pure submodule containing  $M^1$  is essentially finitely indecomposable. After imposing one more condition on  $M$  we obtained some characterizations which shows the relation between center of  $h$ -purity and quasi-essential submodule. In the end of this chapter we discussed the concept of minimal  $h$ -pure submodules of QTAG-modules. We obtained a necessary and sufficient condition for an  $h$ -pure submodules to be a minimal  $h$ -pure submodule containing a given submodule (Theorem 2.3.4).

Chapter 3 devoted to the study of  $h$ -purifiable submodules of QTAG-modules and obtained the relation between purifiability of a submodules and quasi  $h$ -pure submodules. Here we discussed the role of  $h$ -pure and  $h$ -dense submodules of a submodule of a QTAG-module and obtained results which shows that the  $h$ -purifiability of a submodule is very much dependent on the  $h$ -purifiability of  $h$ -pure and  $h$ -dense submodule of the given submodule. We have also established a necessary and sufficient condition for a submodule to be  $h$ -purifiable submodule (Proposition 3.3.5). In the end of this chapter we study about the maximal quasi  $h$ -pure submodule of a QTAG-module and established different characterizations of quasi  $h$ -pure submodules to be maximal quasi  $h$ -pure submodules and obtained different consequences. We also proved that if  $N$  is quasi  $h$ -pure in  $M$  then every submodule  $K$  of  $M$  such that  $N \subseteq K \subseteq \overline{N}$  is also quasi  $h$ -pure (Theorem 3.4.8).

Chapter 4 deals with the study of center of  $h$ -purity. The concept of center of  $h$ -purity was given by Khan [18]. Here we generalize some of results of Ried [41] and Pierce [40] for QTAG-modules. After that we have introduced the concept of subsocles and their interesting properties about height and range. We introduce open subsocles of a QTAG-module and studied some results. In section 4 of this chapter we define a new concept of  $n$ - $h$ -purity, where  $n$  is a non-negative integer. The concept of  $n$ - $h$ -purity generalizes the concept of  $h$ -purity. It is evident that if  $n = 0$  then  $n$ - $h$ -purity is simply  $h$ -purity. We have established that A subsocle  $S$  of a QTAG-module  $M$  becomes center of  $n$ - $h$ -purity if and only if  $h(S) = \infty$ , or  $S$  is open such that  $\text{range}(S) \leq n + 2$  (Theorem 4.4.6).

In section 5 of this chapter we discuss about a special type of QTAG-module obtained by laying down some restrictions on heights of elements of QTAG-module and some characterizations in this regard has been obtained.

In the end, a comprehensive bibliography with author's name in alphabetical order is given enlisting books and papers which have been referred to in this thesis.

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Dated- 30/08/2011

### **Certificate**

*This is to certify that the contents of this thesis entitled “A study of h-pure submodules” has been completed by Miss. Gargi Varshney under my supervision and guidance. She has fulfilled the prescribed conditions given in the statutes and ordinances of Aligarh Muslim University, Aligarh.*

*I further certify that the work of this thesis either partially or fully has not been submitted to any other University or Institution for the award of any degree.*

  
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(**GARGI VARSHNEY**)



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## PREFACE

It is well known that pure subgroups, neat subgroups, basic subgroups, high subgroups, large subgroups and divisible groups etc. have become most useful tools in abelian groups. Most of these concepts have been generalized for modules by different Mathematicians. These studies were carried by imposing some restrictions either on modules or the ring involved. More often, the latter type of study has been made. For instance, different mathematician, Kaplansky, R. Baer, E. Matlis, J.C. Robson, H. Marubayashi etc., studied modules over PID, discrete valuation ring, Dedekind prime ring, generalized uniserial rings etc. This work has been mainly influenced by outstanding researches done by P. Hill, C. Megibben, J.M. Irwin, E.A. Walker, Khalid Ben Abdullah, K. Honda, J.D. Moore, S. Singh and M.Z. Khan.

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# CHAPTER - 1

## PRELIMINARIES

### Section-1

#### § 1.1. Introduction

The principal purpose of this introductory chapter is to recall some necessary definitions, notations and other back ground informations needed for the subsequent chapters. This is being done only to fix up the terminology and notations for subsequent use, and no originality is claimed. The concept of pure subgroups, neat subgroups, divisible subgroups, basic subgroups and high subgroups are quite important objects in abelian groups. Most of these concepts have been generalized by R.B. Warfield [48], H. Marubayashi [37,38] and S. Singh [44,45,46] etc. for modules. Later on Khan [15,16,17,18,19,20,21,22,23] generalized various results for a special type of module, which is called  $S_2$ -module. S. Singh [45] called them TAG-modules and proved that the results which are true for TAG-modules are also true for QTAG-modules. In section 2, some definitions and elementary properties of  $S_2$ -module and QTAG-module have also been given. In section 3, we have given some very useful definitions and results on  $h$ -pure,  $h$ -neat and high submodules as done in [15,17,18,22,45]. In section 4, we have recalled some of the results of  $h$ -divisible and basic submodules from [20,19]. Section 5 deals with the some characterizations of  $h$ -pure submodules as done in [23] and [32]. This chapter is thus only intended to make the thesis as much self contained as possible.

Throughout the thesis we shall consider all the rings  $R$  as associative with unity and the modules as torsion and unital right  $R$ -modules.

## Section-2

### § 1.2. Some Elementary Concepts

**Definition 1.2.1:** A module  $M_R$  is called simple if  $M$  has no proper submodules.

**Definition 1.2.2:** Let  $M_R$  be a module, then the sum of all simple submodules of  $M$  is called socle of  $M$  and is denoted by  $Soc(M)$ .

It is easy to see that for any submodule  $K$  of  $M$ ,  $Soc(K) = K \cap Soc(M)$  and  $Soc(Soc(M)) = Soc(M)$ .

**Proposition 1.2.3 [5, Page 121]:** If  $\{M_\alpha\}_{\alpha \in \Delta}$  is an indexed set of submodule of  $M$  with  $M = \oplus \sum_{\alpha \in \Delta} M_\alpha$  then  $Soc(M) = \oplus \sum_{\alpha \in \Delta} Soc(M_\alpha)$ .

**Definition 1.2.4:** Let  $M$  be a module, then a submodule of  $Soc(M)$  is called sub-socle of  $M$ .

**Definition 1.2.5:** Let  $N$  be a submodule of  $M$ , then  $N$  is called essential submodule of  $M$  if  $N \cap T \neq 0$  for every non-zero submodule  $T$  of  $M$ . It is denoted by  $N \subseteq' M$ .

**Definition 1.2.6:** A module  $M$  extending  $N$  is called an essential extension provided every non-zero submodule of  $M$  has non-zero intersection with  $N$ . In other words if  $N \subset M$ ,  $M$  is an essential extension of  $N$  if and only if  $N$  is essential submodule of  $M$ .

**Proposition 1.2.7:** If  $N$  is essential submodule of  $M$ , then  $Soc(N) = Soc(M)$ .

**Definition 1.2.8:** If  $N$  and  $K$  are submodules of a module  $M$ , then  $N$  is called a

complement of  $K$  if  $N$  is maximal with respect to the property  $N \cap K = 0$ .

**Definition 1.2.9:** A submodule  $T$  of  $M$  is called complement submodule if  $T$  is a complement of some submodule  $U$  of  $M$ .

**Definition 1.2.10:** A submodule  $N$  of  $M$  is called closed in  $M$  if  $N$  has no proper essential extension in  $M$ .

**Definition 1.2.11:** A submodule  $N$  of  $M$  is called direct summand of  $M$  if there exists a submodule  $K$  of  $M$  such that  $M = N \oplus K$ ,  $K$  is called the complementary summand of  $M$ .

**Definition 1.2.12:** A submodule  $N$  of  $M$  is called absolute direct summand of  $M$  if for every complement  $K$  of  $N$  in  $M$ ,  $M = N \oplus K$ .

**Definition 1.2.13:** A module  $M$  is called uniform if intersection of any two of its non-zero submodule is non-zero.

**Definition 1.2.14:** Let  $M$  be a non-zero module. Then a finite chain of submodules of  $M$ ,  $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$  is called a composition series of length  $n$  for  $M$  provided  $M_{i-1}/M_i$  is simple for every  $i$ . If the length of a module  $M$  is  $n$ , then we write  $d(M) = n$ .

**Definition 1.2.15:** A module  $M$  is called uniserial if it has a unique composition series of finite length.

From the definition it follows that uniserial modules are totally ordered.



**Definition 1.2.16:** A module  $M$  is said to be decomposable if it is a direct sum of uniserial modules.

**Definition 1.2.17:** A torsion module  $M$  is called indecomposable if it is not a direct sum of any two of its non-zero submodules.

**Definition 1.2.18:** Let  $N$  be a submodule of  $M$  then  $\{r \in R \mid xr = 0 \text{ for every } x \in N\}$  is called annihilator of  $N$  and is denoted by  $\text{ann}(N)$ .

**Definition 1.2.19:** A module  $M$  is called divisible if  $Mr = M$  for all regular elements  $r \in R$ .

**Definition 1.2.20:** A module  $M$  is called projective if given any diagram,

$$\begin{array}{ccccc} & & M & & \\ & \swarrow h & \downarrow g & & \\ A & \xrightarrow{f} & B & \longrightarrow & 0 \end{array}$$

of  $R$ -modules with exact row, it is always possible to find an  $R$ -homomorphism  $h : M \longrightarrow A$  such that  $f \circ h = g$ .

**Definition 1.2.21:** A module  $M$  is called injective if given any diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow g & & \swarrow h \\ & & M & & \end{array}$$

of  $R$ -modules with exact row, it is always possible to find an  $R$ -homomorphism  $h : B \longrightarrow M$  such that  $h \circ f = g$ .

**Definition 1.2.22:** The minimal injective right  $R$ -module  $E$  containing  $M$  is called injective hull of  $M$  and is denoted by  $E(M)$ .

**Remark 1.2.23:** If  $E$  is the injective hull of  $M$  then  $Soc(M) = Soc(E)$ .

**Definition 1.2.24:** A module  $M$  satisfies ascending chain condition (a.c.c) [descending chain condition (d.c.c)] if every properly ascending (descending) chains of submodules of  $M$  terminates after a finite number of steps.

**Definition 1.2.25:** A module  $M$  is called Noetherian (Artinian) if every ascending (descending) chain of submodules becomes stationary after a finite number of steps.

**Definition 1.2.26:** A subset  $\{x_1, x_2, \dots, x_m\}$  of a module  $M$  is called linearly independent set if  $\sum x_i r_i = 0, r_i \in R$  implies  $x_i r_i = 0, i = 1, \dots, m$ . An infinite subset  $A$  of  $M$  is linearly independent if and only if every finite subset of  $A$  is linearly independent.

Now we shall define some different types of rings.

**Definition 1.2.27:** A ring  $R$  in which every strictly descending chain of right (left) ideals is finite is called right (left) artinian ring.

**Definition 1.2.28:** A ring  $R$  is called right (left) hereditary if every right (left) ideal is projective.

**Definition 1.2.29:** A ring  $R$  is called hereditary if it is both right as well as left hereditary.

**Definition 1.2.30:** A ring  $R$  is called prime ring if  $(0)$  is a prime ideal.

**Definition 1.2.31:** A prime ring  $R$  which is right hereditary, left hereditary, right noetherian and left noetherian is called  $(hnp)$ -ring.

**Definition 1.2.32:** A ring  $R$  is called right (left) bounded if each of its essential right (left) ideal contains a non-zero two sided ideal.

**Definition 1.2.33:** In a module  $M$ , an element  $x$  is said to be a torsion element if  $xr = 0$  for some regular element  $r \in R$ . The set of all torsion elements  $T(M)$  forms a submodule and is called torsion submodule of  $M$ . A module  $M$  is said to be torsion module if  $T(M) = M$ . Equivalently if every non-zero element  $M$  is torsion.

**Proposition 1.2.34 [Lemma 1 & 2, 42]:** Let  $R$  be a bounded  $(hnp)$ -ring then the following hold:

- (a) Every finitely generated torsion  $R$ -module is a direct sum of finitely many uniserial modules.
- (b) Any uniform torsion  $R$ -module is either of finite length and uniserial or is injective and of finite length.
- (c) Let  $U$  and  $V$  be two uniform torsion right  $R$ -modules and  $b(\neq 0) \in U$ . If  $f : bR \rightarrow V$  is a non-zero  $R$ -homomorphism and  $\text{length}, d(U/bR) \leq d(V/f(bR))$ , then  $f$  can be extended to an  $R$ -homomorphism  $g : U \rightarrow V$  and  $U/bR \cong g(U)/g(bR)$ .
- (d) Any non-zero homomorphic image of a uniform, torsion  $R$ -module is uniform.

Let  $R$  be an associative ring with identity and  $M$  be a unital right  $R$ -module. Consider the following conditions of  $M_R$  as introduced by Singh [44].

- (I) Every finitely generated submodule of every homomorphic image of  $M$  is a direct sum of uniserial modules.
- (II) Given any two uniserial submodules  $U$  and  $V$  of a homomorphic image of  $M$ , for any submodule  $W$  of  $U$ , any non-zero homomorphism  $f : W \rightarrow V$  can be extended to a homomorphism  $g : U \rightarrow V$  provided the composition length  $d(U/W) \leq d(V/f(W))$ .

**Definition 1.2.35 [22]:** A module  $M$  satisfying condition (I) and (II) is called an  $S_2$ -module.

**Definition 1.2.36 [45]:** A module  $M$  satisfying only the condition (I) is said to be the QTAG-module.

Now we give some elementary definitions and results as introduced in [15,17,22,45].

**Definition 1.2.37:** Let  $M$  be an  $S_2$ -module, then an element  $x(\neq 0)$  of  $M$  is called uniform if  $xR$  is a uniform module.

**Definition 1.2.38:** Let  $M$  be an  $S_2$ -module, then an uniform element  $x \in M$  is called of exponent  $n$  (denoted by  $e(x)$ ) if  $d(xR) = n$ ; and the  $\sup\{d(yR/xR)/yR$  is uniserial submodule of  $M$  containing  $x\}$  is called the height of  $x$  and is denoted by  $H_M(x)$  (or simply  $H(x)$ ).

**Definition 1.2.39:** An  $S_2$ -module  $M$  is called bounded if there exists a positive integer  $k$  such that  $H(x) \leq k$  for all uniform elements  $x \in M$ .

**Proposition 1.2.40 [Lemma 4, 42]:** Let  $M$  be a module and  $x_1, x_2, \dots, x_n$  be finitely many uniform element of  $M$  such that for some positive integer  $k$ ,  $H(x_i) \geq k$  for all  $i$ . Then for every uniform element  $x$  of  $M$  in  $\sum_i x_i R$ ,  $H(x) \geq k$ .

**Definition 1.2.41:** Let  $M$  be an  $S_2$ -module, then for every  $k \geq 0$ ,  $H_k(M)$  will denote the submodule of  $M$  generated by the uniform elements of  $M$ , which are of height  $\geq k$ .

**Definition 1.2.42:** Let  $M$  be a  $S_2$ -module, then for every  $k \geq 0$ ,  $H^k(M)$  will denote the submodule of  $M$  generated by the uniform elements of exponent at most  $k$ .

**Definition 1.2.43:** Let  $N$  be a submodule of  $S_2$ -module  $M$  then for any integer  $k \geq 0$ , we define  $H^k(N)$  to be submodule of  $M$  generated by those elements  $x \in M$  for which the elements  $\bar{x} = x + N$  in  $M/N$  has exponent  $\leq k$ .

In other words  $H^k(N)$  is the submodule generated by those uniform elements  $x \in M$  for which  $d(xR/(xR \cap N)) \leq k$  i.e. there exists at least a uniform element  $y \in xR \cap N$  such that  $d(xR/yR) \leq k$  and we denote  $H_N^n(0)$  by  $Soc^n(N)$ .

**Proposition 1.2.44 [Corollary 1, 44]:** Any bounded  $S_2$ -module  $M$  is a direct sum of uniserial modules.

**Proposition 1.2.45 [Lemma 6, 42]:** Let  $M = A + B$  be a torsion  $R$ -module and  $A, B$  be its submodules. Then for any non-negative integer  $k$ ,  $H_k(M) = H_k(A) + H_k(B)$ .

**Lemma 1.2.46 [Lemma 2.3, 45]:** Let  $A$  and  $B$  be any two uniserial submodules

of a QTAG-module  $M$  such that  $A \cap B \neq 0$  and  $d(A) \leq d(B)$ . Then there exists a monomorphism  $\sigma : A \longrightarrow B$ , which is identity on  $A \cap B$ .

**Lemma 1.2.47 [Lemma 3.9, 45]:** Let  $N$  be a submodule of a QTAG-module  $M$ . Then  $N$  is an  $h$ -pure submodule of  $M$  if and only if for every uniform element  $\bar{x} = x + N$  of  $M/N$ , there exists a uniform element  $x' \in M$  such that  $x + N = x' + N$  and  $e(x') = e(x)$ .

**Theorem 1.2.48 [Theorem 3.11, 45]:** (a) If every element in  $Soc(M)$  is of infinite height, then  $M$  is a direct sum of serial modules, each of infinite length.  
(b) Any QTAG-module  $M$  admits a uniform summand, which can be chosen to be of finite length in case not all uniform element in  $Soc(M)$  are of infinite heights.

## Section-3

### § 1.3. $h$ -pure and $h$ -neat Submodules

This section is significant in the sense that some of the result mentioned here have been very often used in the subsequent chapters. As it is obvious from the heading of this section,  $h$ -pure,  $h$ -neat and high submodules are given here.

**Proposition 1.3.1 [Theorem 2, 44]:** Let  $M$  be an  $S_2$ -module and  $N$  be a submodule of  $M$  such that  $N$  is a direct sum of uniserial modules of same length  $k$ . Then the following are equivalent:

- (a)  $N$  is a direct summand of  $M$ .
- (b)  $H_n(N) = N \cap H_n(M)$  for all  $n$ .
- (c)  $N$  satisfies  $H_k(M) \cap N = 0$ .

**Definition 1.3.2 [44]:** A submodule  $N$  of an  $S_2$ -module  $M$  is called  $h$ -pure if  $H_k(N) = N \cap H_k(M)$  for all non-negative integer  $k$ .

**Proposition 1.3.3:** Every direct summand of an  $S_2$ -module is  $h$ -pure.

**Proposition 1.3.4 [Theorem 3, 44]:** Every bounded  $h$ -pure submodule of an  $S_2$ -module  $M$  is a direct summand of  $M$ .

**Proposition 1.3.5 [Lemma 1, 44]:** Let  $x$  be a uniform element in  $\text{Soc}(M)$  such that  $H(x)$  is finite. If  $u \in M$  is a uniform element such that  $x \in uR$  and  $d(uR/xR) = H(x)$ , then  $uR$  is a  $h$ -pure submodule of  $M$  and hence a summand of  $M$ .

**Proposition 1.3.6 [Lemma 2, 44]:** Let  $N$  be a submodule of an  $S_2$ -module  $M$ , then the following hold:

- (i) If  $N$  is  $h$ -pure in  $M$ , given any uniform element  $\bar{x} \in M/N$  there exists a uniform element  $x' \in N$  such that  $e(\bar{x}) = e(x')$  and  $\bar{x} = \bar{x}'$ .
- (ii) If  $N$  is  $h$ -pure in  $M$ , and  $K$  is any submodule of  $N$ , then  $N/K$  is  $h$ -pure in  $M/K$ .
- (iii) If  $K$  is  $h$ -pure submodule of  $M$  such that  $K \subseteq N$  and  $N/K$  is  $h$ -pure in  $M/K$ , then  $N$  is  $h$ -pure in  $M$ .

**Proposition 1.3.7 [Theorem 4, 44]:** Let  $M$  be an  $S_2$ -module. If every uniform element of  $\text{Soc}(M)$  is of infinite height, then  $M$  is a direct sum of infinite length uniform submodules.

**Proposition 1.3.8 [Proposition 2, 15]:** If  $M$  is a  $S_2$ -module and  $N$  is  $h$ -pure submodule of  $M$  with same socle then  $N = M$ .

**Proposition 1.3.9 [Lemma 2, 15]:** If  $N$  is a  $h$ -pure submodule of a  $S_2$ -module  $M$  such that  $Soc(H_k(M)) \subseteq N$  for some non-negative integer  $k$ , then  $H_k(M) \subseteq N$ .

**Proposition 1.3.10 [Lemma 3, 15]:** If  $K$  is  $h$ -pure submodule of a  $S_2$ -module  $M$  then  $Soc(H_n(M/K)) = (Soc(H_n(M)) + K)/K$ .

**Proposition 1.3.11 [Lemma 2, 22]:** If  $N$  is a submodule of a  $S_2$ -module  $M$  and for every uniform element  $x \in Soc(N)$ ,  $H_N(x) = H_M(x)$ , then  $N$  is  $h$ -pure submodule of  $M$ .

**Proposition 1.3.13 [Proposition 2.5, 31]:** If  $M$  is a  $S_2$ -module such that  $M/K = N/K \oplus T/K$ , where  $N$ ,  $T$  and  $K$  are the submodules of  $M$  and  $K$  is  $h$ -pure in  $N$ , then  $T$  is also  $h$ -pure in  $M$ .

**Definition 1.3.14 [18]:** If  $M$  is a  $S_2$ -module and  $N$  is a submodule of  $M$  then  $N$  is called center of  $h$ -purity in  $M$  if every complement of  $N$  in  $M$  is  $h$ -pure in  $M$ .

**Proposition 1.3.15 [Corollary 5, 18]:** If  $M$  is a  $S_2$ -module then for every  $k \geq 0$ ,  $H_k(M)$  is center of  $h$ -purity.

**Definition 1.3.16 [4]:** If  $M$  is an  $S_2$ -module then a submodule  $S$  of  $Soc(M)$  is called subsocle.

**Definition 1.3.17 [2]:** A subsocle  $S$  of an  $S_2$ -module  $M$  is said to support a submodule  $N$  of  $M$  if  $S = Soc(N)$ .



**Definition 1.3.18** [17]: An  $S_2$ -module is called  $h$ -pure complete if every subsocle of  $M$  supports an  $h$ -pure submodule.

**Definition 1.3.19** [17]: A submodule  $N$  of an  $S_2$ -module  $M$  is called  $h$ -neat if  $H_1(N) = N \cap H_1(M)$ .

**Proposition 1.3.20** [Theorem 3, 17]: A submodule  $N$  of a  $S_2$ -module  $M$  is  $h$ -neat in  $M$  if and only if  $N$  has no proper essential extension in  $M$ .

**Proposition 1.3.21** [Proposition 4, 22]: If  $M$  is an  $S_2$ -module and  $N$  is a submodule of  $M$  then any complement  $T$  of  $N$  is  $h$ -neat.

**Definition 1.3.22** [4]: If  $N$  is a submodule of an  $S_2$ -module  $M$ , then  $h$ -neat hull of  $N$  is defined as the minimal  $h$ -neat submodule  $K$  of  $M$  such that  $N \subseteq K$ .

**Definition 1.3.23** [27]: If  $N$  is a submodule of a QTAG-module  $M$ , then  $N$  is called kernel of  $h$ -purity if  $h$ -neat hulls of  $N$  are  $h$ -pure submodule of  $M$ .

**Definition 1.3.24**: The submodule of  $M$  generated by the uniform elements of infinite height is denoted by  $M^1$ . Equivalently  $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ .

**Definition 1.3.25** [22]: A submodule  $N$  of an  $S_2$ -module  $M$  is called high submodule if it is a complement of  $M^1$ .

**Proposition 1.3.26** [Theorem 7, 22]: If  $N$  is a submodule of an  $S_2$ -module  $M$  such that  $N \subseteq M^1$ . Then any complement  $T$  of  $N$  is  $h$ -pure submodule of  $M$ .

**Proposition 1.3.27** [Proposition 13, 22]: If  $M$  is an  $S_2$ -module and  $N \subseteq M^1 \neq 0$  then for any complement  $T$  of  $N$  in  $M$ ,  $M/T$  is direct sum of infinite length uniform submodules.

**Theorem 1.3.28** [Lemma 4, 1]: Let  $N$  be a submodule of an  $S_2$  module  $M$  such that for some  $n$ ,  $N + H_n(M) \not\supseteq \text{Soc}(H_{n-1}(M))$ . Then there exists a proper  $h$ -pure submodule  $H$  of  $M$  such that  $H \supseteq N + H_n(M)$  and  $N + H_n(M) \supseteq \text{Soc}(H_{n-1}(M))$ .

**Theorem 1.3.29** [Theorem 6, 1]: Let  $K$  be an  $h$ -pure submodule of an  $S_2$  module  $M$ , containing a submodule  $N$  of  $M$ . If  $K$  is a minimal  $h$ -pure submodule of  $M$  containing  $N$ , then  $N + H_n(K) \supseteq \text{Soc}(H_{n-1}(K))$  for all  $n$ . On the converse, if  $N + H_n(K) \supseteq \text{Soc}(H_{n-1}(K))$  for all  $n$  and  $N \supseteq \text{Soc}(H_m(K))$  for some  $m$ , then  $K$  is a minimal  $h$ -pure submodule of  $M$ , containing  $N$ .

## Section-4

### § 1.4. $h$ -divisible and Basic Submodules

In this section, we recall some definitions and properties of  $h$ -divisible and basic submodules for  $S_2$ -modules as introduced by Khan in [20] and [19] respectively.

**Definition 1.4.1** [20]: Let  $M$  be a  $S_2$ -module, then  $M$  is called  $h$ -divisible if  $H_1(M) = M$ .

**Remark 1.4.2:** An  $S_2$ -module  $M$  is  $h$ -divisible if and only if every uniform element of  $M$  is of infinite height.

**Proposition 1.4.3 [Lemma 1, 20]:** Let  $M$  be a  $S_2$ -module and  $M = \oplus \sum M_\alpha$  then  $M$  is  $h$ -divisible if and only if each  $M_\alpha$  is  $h$ -divisible.

**Proposition 1.4.4 [Lemma 2, 20]:** Let  $M$  be a  $S_2$ -module, then  $M$  is  $h$ -divisible if and only if every uniform element of  $\text{Soc}(M)$  is of infinite height.

**Theorem 1.4.5 [Theorem 3, 20]:** If  $M$  is a  $S_2$ -module then  $M$  is  $h$ -divisible if and only if  $M$  is a direct sum of infinite length uniform submodules.

**Theorem 1.4.6 [Theorem 4, 20]:** Let  $M$  be a  $S_2$ -module and  $N$  be a  $h$ -divisible submodule of  $M$  then  $N$  is a direct summand of  $M$ .

**Proposition 1.4.7 [Proposition 6, 28]:** If  $N$  is  $h$ -pure submodule of a QTAG-module  $M$  such that  $M/N$  is  $h$ -divisible then  $\text{Soc}(M) = \text{Soc}(N) + \text{Soc}(H_n(M))$  for all  $n$ .

**Theorem 1.4.8 [Corollary 8, 28]:** If  $M$  is a QTAG-module and  $N$  is a submodule of  $M^1$ , then every complement  $K$  of  $N$  is  $h$ -pure in  $M$  and  $M/K$  is  $h$ -divisible.

**Theorem 1.4.9 [Corollary 10, 29]:** If  $M$  is a QTAG-module and  $N$  is a submodule of  $M$  then  $M/K$  is  $h$ -divisible for every complement  $K$  of  $N$  if and only if  $\text{Soc}(N) \subseteq M^1$ .

**Definition 1.4.10:** Let  $M$  be a QTAG-module. The divisible hull of  $M$  is the intersection of all divisible QTAG-modules containing  $M$ . In other words, it is the smallest divisible QTAG-module containing  $M$ .

**Definition 1.4.11:** An  $S_2$ -module  $M$  is said to be reduced if it is free from the elements of infinite height. Equivalently  $\{0\}$  is the only  $h$ -divisible submodule of  $M$ .

**Definition 1.4.12 [3]:** A submodule  $N$  of an  $S_2$ -module  $M$  is called  $h$ -dense in  $M$  if and only if  $M/N$  is  $h$ -divisible.

**Proposition 1.4.13 [Proposition 2, 3]:** A submodule  $N$  of an  $S_2$ -module  $M$  is  $h$ -dense in  $M$  if and only if  $M = N + H_n(M)$  for all non negative integers  $n$ .

**Definition 1.4.14 [4]:** A submodule  $N$  of an  $S_2$  module  $M$  is said to be almost  $h$ -dense in  $M$  if for every  $h$ -pure submodule  $K$  of  $M$  containing  $N$ ,  $M/K$  is  $h$ -divisible.

**Theorem 1.4.15 [Theorem 5, 1]:** A submodule  $N$  of an  $S_2$  module  $M$  is almost  $h$ -dense in  $M$  if and only if  $N + H_n(M) \supseteq \text{Soc}(H_{n-1}(M))$  for all  $n$ .

**Definition 1.4.16 [19]:** Let  $M$  be an  $S_2$ -module. A submodule  $B$  of  $M$  is called a basic submodule of  $M$  if the following conditions hold:

- (i)  $B$  is a direct sum of uniserial submodules
- (ii)  $B$  is  $h$ -pure in  $M$
- (iii)  $M/B$  is  $h$ -divisible

The following theorem shows the existence of basic submodules.

**Theorem 1.4.17 [Theorem 1, 19]:** Let  $M$  be a QTAG-module then  $M$  possesses a basic submodule.

**Theorem 1.4.18 [Theorem 4, 19]:** A submodule  $N$  of a QTAG-module can be extended to a basic submodule  $B$  of  $M$  if and only if  $N = \cup_i C_i$  where  $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \subseteq \cdots$ , such that the height of uniform elements of  $C_n$  (taken in  $M$ ) are bounded.

The following result gives the uniqueness of basic submodules.

**Theorem 1.4.19 [Theorem 5, 19]:** If  $M$  is a QTAG-module, then any two basic submodules are isomorphic.

**Theorem 1.4.20 [Theorem 4, 30]:** If  $M$  is a QTAG-module, then  $M$  has only one basic submodule if and only if it is either  $h$ -divisible or bounded.

## Section-5

### § 1.5. Some Characterizations of $h$ -pure Submodules

Here we state some important definitions and results of [23] and [32] which are general in nature but significant for next chapters.

**Definition 1.5.1 [23]:** Let  $M$  be an  $S_2$ -module and  $N$  be a submodule of  $M$ , then  $N$  is called quasi-essential in  $M$  if  $M = T + K$  where  $K$  is complement of  $N$  and  $T$  is any  $h$ -pure submodule of  $M$  containing  $N$ .

Let  $M$  be an  $S_2$ -module satisfying the following condition introduced by Singh (unpublished):

(A) For any finitely generated submodule  $N$  of  $M$ ,  $R/ann(N)$  is right artinian.

**Theorem 1.5.2 [Corollary 2, 23]:** Let  $M$  be a  $S_2$ -module,  $N$  be a submodule of  $M^1$  and  $K$  be any  $h$ -pure submodule of  $M$  containing  $N$ . Then for any complement  $T$  of  $N$  in  $M$ ,  $M = T + K$ .

**Theorem 1.5.3 [Corollary 8, 23]:** Let  $M$  be an  $S_2$ -module satisfying the condition (A) and  $S \subseteq \text{Soc}(M)$  such that  $\text{Soc}(H_n(M)) \supseteq S \supseteq \text{Soc}(H_{n+1}(M))$  for some  $n$ , then  $S$  is quasi-essential in  $M$ .

**Theorem 1.5.4 [Theorem 12, 23]:** Let  $M$  be an  $S_2$ -module satisfying the condition (A) and  $S \subseteq \text{Soc}(M)$  such that  $S \not\subseteq M^1$ , then the following are equivalent:

- (i)  $S$  is both a center of  $h$ -purity in  $M$  and a quasi-essential subsocle of  $M$ .
- (ii)  $S$  supports an absolute summand.
- (iii) There exists a natural number  $n$  such that  $\text{Soc}(H_n(M)) \supseteq S \supseteq \text{Soc}(H_{n+1}(M))$ .

**Notation 1.5.5 [32]:** For any non-negative integer  $t$  and for a submodule  $N$  of a QTAG-module  $M$ , we denote by  $N^t(M)$  the submodule  $(N + H_{t+1}(M)) \cap \text{Soc}(H_t(M))$  and by  $N_t(M)$  the submodule  $(N \cap \text{Soc}(H_t(M))) + \text{Soc}(H_{t+1}(M))$  and by  $Q_t(M, N) = N^t(M)/N_t(M)$ .

**Definition 1.5.6 [32]:** A submodule  $N$  of a QTAG-module  $M$  is quasi  $h$ -pure in  $M$  if  $Q_n(M, N) = 0$  for all  $n \geq 0$ .

**Proposition 1.5.7 [Proposition 4.5, 32]:** If  $N$  is  $h$ -pure submodule of  $M$  of if  $N$  is a subsocle of  $M$ , then  $N$  is quasi  $h$ -pure.

**Theorem 1.5.8 [Theorem 4.6, 32]:** If  $N$  is a submodule of a QTAG-module  $M$ , then the following are equivalent:

- (a)  $N$  is quasi  $h$ -pure in  $M$
- (b)  $Soc(N + H_n(M)) = Soc(N) + Soc(H_n(M))$  for all  $n \geq 1$
- (c)  $H_1(N \cap H_n(M)) = H_1(N) \cap H_{n+1}(M)$  for all  $n \geq 1$

**Theorem 1.5.9 [Theorem 4.7, 32]:** If  $N$  is a submodule of a  $M$ , then  $N$  is  $h$ -pure in  $M$  if and only if  $N$  is  $h$ -neat and quasi  $h$ -pure in  $M$ .

**Theorem 1.5.10 [Theorem 4.12, 32]:** If  $K$  is  $h$ -pure submodule of  $H_n(M)$ , where  $n \geq 0$ . Then every submodule  $T$  of  $M$  maximal with respect to  $T \cap H_n(M) = K$ , is  $h$ -pure in  $M$ .

## CHAPTER - 2

### Quasi-Essential Submodules and Minimal $h$ -pure Submodules of QTAG-module

#### Section-1

##### § 2.1. Introduction

The concept of quasi-essential submodules has been studied in [23] and different characterizations were obtained in terms of center of  $h$ -purity. A submodule  $N$  of a QTAG-module  $M$  is called quasi-essential if  $M = T + K$  for a complement  $K$  of  $N$  and  $T$  an  $h$ -pure submodule of  $M$  containing  $N$ .

In section 2, we extend the study of quasi-essential submodules. First of all we generalize a theorem of L. Fuchs [11], which is of very interesting nature. Here we characterize quasi-essential submodules i.e. We proved that a submodule  $N$  of a QTAG-module  $M$  is quasi-essential if and only if  $K/T$  is an absolute summand of  $M/T$  where  $K$  is an  $h$ -pure submodule of  $M$  containing  $N$  and  $T$  is a complement of  $K$  (Proposition 2.2.10). Further we established various conditions under which  $h$ -pure submodules are direct summands. We also introduced the concept of essentially finitely indecomposable QTAG-module and prove that every  $h$ -pure submodule containing  $M^1$  is essentially finitely indecomposable. In the end of this section after imposing one more condition on  $M$  many results have been proved to see the relation between center of  $h$ -purity and quasi-essential submodules. It has been seen in [23] that all subsocles of  $M^1$  are quasi-essential and condition has been obtained under which every quasi-essential subsocle is center of  $h$ -purity. So here we obtain a similar characterization.



Section 3 is devoted to the study of minimal  $h$ -pure submodules of QTAG-modules. In this section we obtained a necessary and sufficient condition for an  $h$ -pure submodule to be a minimal  $h$ -pure submodule containing a given submodule (Theorem 2.3.4). Further we prove that a minimal  $h$ -pure submodule containing a submodule of a basic submodule of a QTAG-module becomes a direct sum of uniserial submodules (Theorem 2.3.5).

## Section-2

### § 2.2. Quasi-Essential Submodules

First of all we restate the following result from [27].

**Lemma A:** If  $A$  and  $B$  are any two uniserial submodules of a QTAG-module  $M$  such that  $A \cap B \neq 0$  and  $d(A) \leq d(B)$ . Then there exists a monomorphism  $\sigma : A \longrightarrow B$ , which is identity on  $A \cap B$ .

**Proof:** As  $d(A) \leq d(B)$ ,  $A + B = B \oplus C$ . Now the restriction of the projection  $\rho : B \oplus C \longrightarrow B$ , to  $A$  is a desired map.

Now we generalize [Theorem 66.3, 11], which itself is of interesting nature.

**Theorem 2.2.1:** If  $M$  is a QTAG-module then every  $h$ -dense subsocle of  $M$  supports an  $h$ -pure and  $h$ -dense submodule.

**Proof:** Let  $S$  be a subsocle of  $M$  and  $S$  be  $h$ -dense; then  $Soc(M) = S + Soc(H_k(M))$  for all  $k \in \mathbb{Z}^+$ . Let  $N$  be maximal with the property  $Soc(N) = S$ . Firstly we show that  $N$  is  $h$ -neat submodule of  $M$ . Let  $x$  be a uniform element in  $N \cap H_1(M)$ ,

then for a uniform element  $y \in M$ , we have  $d(yR/xR) = 1$ . If  $y \in N$ , then  $x \in H_1(N)$ . Let  $y \notin N$  then  $S \subsetneq Soc(N + yR)$ . Hence, there exists a uniform element  $z \in Soc(N + xR)$  such that  $z \notin S$  and  $z = u + yr$  where  $u \in N$  and  $r \in R$ . Trivially  $yrR = yR$ , hence without any loss of generality we can assume  $z = u + y$ . Define a map  $\eta : yR \rightarrow uR$  such that  $\eta(yr) = ur$ . Let  $yr = 0$ , then  $zr = ur$ . If  $zrR = zR$  then  $z \in S$ , a contradiction, therefore  $zr = 0$  and we get  $ur = 0$ , consequently  $\eta$  is a well defined epimorphism. Therefore,  $uR$  is a uniform submodule. Since  $u + y \in Soc(M)$ ,  $H_1(uR) = H_1(yR)$ , but  $xR$  is a maximal submodule of  $M$ ; hence  $H_1(yR) = xR$  and we get  $x \in H_1(N)$ . Thus,  $N \cap H_1(M) = H_1(N)$ . Now suppose  $N \cap H_n(M) = H_n(N)$  and let  $x$  be a uniform element in  $N \cap H_{n+1}(M)$ ; then  $d(yR/xR) = 1$  for some uniform element  $y \in H_n(M)$ . Since  $N$  is  $h$ -neat in  $M$ , there is a uniform element  $y' \in N$  such that  $d(y'R/xR) = 1$ . Hence by Lemma A, there exists an isomorphism  $\sigma : yR \rightarrow y'R$  which is identity on  $xR$ . The map  $\eta : yR \rightarrow (y - y')R$  where  $\sigma(y) = y'$  is an epimorphism with  $xR \subseteq \text{Ker } \eta$ . Hence,  $e(y - y') \leq 1$  and we get  $y - y' \in Soc(M) = S + Soc(H_n(M))$ . Therefore,  $y - y' = s + t$  for some  $s \in S, t \in H_n(M)$ . Consequently,  $y - t = y' + s \in N \cap H_n(M) = H_n(N)$ . Since  $y - y' - s \in Soc(M)$ ,  $H_1(yR) = H_1((y' + s)R) \subseteq H_{n+1}(N)$ . Hence,  $x \in H_{n+1}(N)$ . Therefore,  $N$  is  $h$ -pure submodule of  $M$ .

Now let  $\bar{x} \in Soc(M/N) = (Soc(M) + N)/N$  be a uniform element; then by Lemma 1.2.47 there exists a uniform element  $x' \in M$  such that  $\bar{x} = \bar{x}'$  and  $e(x') = 1$ . Since  $Soc(M) = S + Soc(H_k(M))$  for all  $k$ , we get  $\bar{x} \in H_k(M/N)$  for every  $k$ . Hence,  $\bar{x} \in \cap_{k=1}^{\infty} H_k(M/N)$  and appealing to Theorem 1.2.48, we get  $M/N$  is  $h$ -divisible. Hence,  $N$  is  $h$ -dense in  $M$ .

Now we state the following lemmas. Since their proofs are of set theoretic nature, therefore the same is omitted.

**Lemma 2.2.2:** If  $M$  is QTAG-module and  $K \subseteq N \subseteq M$  and  $T$  is a complement of  $K$  then  $T \cap N$  is complement of  $K$  in  $N$ . Conversely, if  $L$  is complement of  $K$  in  $N$ , then  $L = T \cap K$  whenever  $T$  is complement of  $K$  of  $M$  containing  $L$ .

**Lemma 2.2.3:** If  $M$  is QTAG-module and  $K \subseteq N \subseteq M$ . If  $T$  is a complement of  $K$ , then every complement of  $T \cap N$  in  $T$  is a complement of a complement of  $N$  in  $M$ .

**Lemma 2.2.4:** If  $M$  is QTAG-module and  $K \subseteq N \subseteq M$  and  $T$  is a complement of  $K$  in  $N$ . Then a submodule  $L$  containing  $T$  is a complement of  $K$  in  $M$  if and only if  $L/T$  is a complement of  $N/T$  in  $M/T$ .

**Lemma 2.2.5:** If  $M$  is QTAG-module and  $N, K$  are submodules of  $M$  such that  $N \cap K = 0$ , then a submodule  $T$  containing  $K$  is a complement of  $N$  in  $M$  if and only if  $T/K$  is a complement of  $(N \oplus K)/K$  in  $M/K$ .

Now we prove few lemmas which are used later and are of independent interest.

**Lemma 2.2.6:** If  $M$  is QTAG-module and  $K \subseteq N \subseteq T$  are submodules of  $M$  and  $N$  is an  $h$ -pure submodules of  $M$ . Then  $T/K$  is  $h$ -pure in  $M/K$  if and only if  $T$  is  $h$ -pure in  $M$ .

**Proof:** If  $T$  is  $h$ -pure in  $M$  then trivially  $T/K$  is  $h$ -pure in  $M/K$ .

Conversely, let  $T/K$  be  $h$ -pure in  $M/K$  and let  $f$  be the canonical map defined as  $f : M/K \longrightarrow M/N$  such that  $f(x + K) = x + N$  then  $\text{Ker } f \subseteq T/K$  and  $f(T/K) = T/N$ , therefore  $T/N$  is  $h$ -pure in  $M/N$ . Since  $N$  is  $h$ -pure in  $M$ , so  $T$  is  $h$ -pure in  $M$ .

**Lemma 2.2.7:** If  $M$  is QTAG-module,  $N$  is a submodule of  $M$  and  $B$  is an  $h$ -pure,  $h$ -dense submodule of  $N$ . Then there exists an  $h$ -pure,  $h$ -dense submodule  $K$  of  $M$  such that  $K \cap N = B$ .

**Proof:** Since  $B$  is  $h$ -dense in  $N$ , we have  $M/B = N/B \oplus K/B$  for some submodule  $K$  of  $M$ , then by Proposition 1.3.13,  $K$  is  $h$ -pure in  $M$  and trivially  $K \cap N = B$ .

**Proposition 2.2.8:** Let  $M$  be a QTAG-module and  $S$  be a subsocle of  $Soc(M)$  such that  $S \not\subseteq M^1$ . Let  $K$  be a maximal  $h$ -pure submodule of  $M$  such that  $Soc(K) \subseteq S$ . Then  $(S + K)/K$  is contained in the  $h$ -reduced part of  $(M/K)^1$ .

**Proof:** Trivially  $S$  has at least one element of finite height, therefore, there exists at least one  $h$ -pure submodule  $T$  of  $M$  such that  $Soc(T) \subseteq S$ . The existence of a maximal element is ensured by Zorn's Lemma, therefore we get a maximal  $h$ -pure submodule  $K$  of  $M$  such that  $Soc(K) \subseteq S$ . Trivially  $(S + K)/K \subseteq Soc(M/K)$ . If  $(S + K)/K$  has an element of finite height then  $M/K = K'/K \oplus L/K$  such that  $Soc(K'/K) \subseteq (S + K)/K$ , hence  $Soc(K') \subseteq S$  and since  $K'$  is  $h$ -pure in  $M$ , we get a contradiction to the maximality of  $K$ .

Therefore,  $(S + K)/K \subseteq (M/K)^1$ . Since  $h$ -divisible submodules are absolute summands, hence we ultimately get  $(S + K)/K$  contained in the  $h$ -reduced part of  $(M/K)^1$ .

**Proposition 2.2.9:** If  $M$  is a QTAG-module such that  $M = B \oplus D$  where  $B$  is bounded and  $D$  is  $h$ -divisible, then every  $h$ -pure submodule  $K$  of  $M$  is the direct sum of bounded and  $h$ -divisible submodule.

**Proof:** Let  $M = B \oplus D$  where  $B$  is bounded and  $D$  is  $h$ -divisible. Let  $K$  be an  $h$ -pure submodule of  $M$ , then  $K \cap D = K^1$ . Let  $T$  be a complement of  $K^1$  in  $K$ , then  $T \cap D = 0$  and therefore  $T$  is bounded. Hence,  $K = T \oplus (K \cap D)$  where  $(K \cap D) \cong K/T$  is  $h$ -divisible.

**Proposition 2.2.10:** If  $M$  be a QTAG-module and  $N \subseteq M$ , then  $N$  is quasi-essential submodule of  $M$  if and only if  $K/T$  is an absolute summand of  $M/T$  whenever  $K$  is an  $h$ -pure submodule of  $M$  containing  $N$  and  $T$  is a complement of  $K$ .

**Proof:** Let  $A/T$  be a complement of  $K/T$  in  $M/T$ , then by Lemma 2.2.5,  $A$  is a complement of  $N$  and if  $N$  is quasi-essential, then we get  $M = A + K$ . Therefore,  $M/T = A/T \oplus K/T$ .

Conversely, let  $A$  be a complement of  $N$  in  $M$ , then by Lemma 2.2.3,  $A \cap K$  is a complement of  $N$  in  $K$ . Hence,  $K/(A \cap K)$  is an absolute summand of  $M/(A \cap K)$  and by Lemma 2.2.5,  $A/(A \cap K)$  is a complement of  $K/(A \cap K)$  in  $M/(A \cap K)$ . Therefore,  $M/(A \cap K) = A/(A \cap K) \oplus K/(A \cap K)$  and we get  $M = A + K$ . Therefore,  $N$  is quasi-essential submodule of  $M$ .

**Theorem 2.2.11:** If  $M$  is a QTAG-module and  $S$  is a submodule of  $M^1$ . Then every  $h$ -pure submodule of  $M$  containing  $S$  is summand of  $M$  if and only if  $M$  is a direct sum of a bounded submodule and  $h$ -divisible submodule.

**Proof:** Let  $K$  be a complement of  $M^1$ , then  $K$  is  $h$ -pure and  $M/K$  is  $h$ -divisible Proposition 1.3.26 and Proposition 1.3.27. If  $K$  is unbounded then  $K$  contains a proper basic submodule  $B$  of  $K$  and hence  $M/B = K/B \oplus T/B$  where  $T$  can be

chosen to contain  $M^1$  as  $K \cap M^1 = 0$ . Appealing to Proposition 1.3.13,  $T$  is an  $h$ -pure submodule of  $M$  and  $S \subseteq T$ . Therefore,  $M = T \oplus A$  and  $A$  is  $h$ -divisible, which is a contradiction. Hence,  $K$  is bounded and therefore  $K$  is a summand of  $M$  i.e.  $M = K \oplus D$  where  $D$  is  $h$ -divisible.

For the converse we refer to Proposition 2.2.9.

**Theorem 2.2.12:** If  $M$  is a QTAG-module and  $S$  is a subsocle of  $M$ . Then the following are equivalent:

- (i)  $S \supseteq \text{Soc}(M^1)$  and every  $h$ -pure submodule of  $M$  containing  $S$  is a summand of  $M$ .
- (ii) Every  $h$ -pure submodule of  $M$  containing  $S$  is a cobounded summand of  $M$ .
- (iii)  $S \supseteq \text{Soc}(H_n(M))$ , for some positive integer  $n$ .

**Proof:** We establish (ii)  $\rightarrow$  (i)  $\rightarrow$  (iii)  $\rightarrow$  (ii)

(ii)  $\rightarrow$  (i) Let  $x$  be a uniform element in  $\text{Soc}(M^1)$  and  $x \notin S$ , then  $xR \cap S = 0$ . Embedding  $S$  into a complement  $K$  of  $xR$ . Then  $K$  is an  $h$ -pure submodule of  $M$  and  $M/K$  is  $h$ -divisible, which is a contradiction. Therefore,  $x \in S$  and we get  $\text{Soc}(M^1) \subseteq S$ .

(i)  $\rightarrow$  (iii) Let  $S = M^1$ , then by Theorem 2.2.11,  $M = B \oplus D$  where  $B$  is bounded and  $D$  is  $h$ -divisible. Let  $H_n(B) = 0$ , then clearly  $\text{Soc}(H_n(M)) \subseteq S$ . Let  $S \neq M^1$  and  $K$  be a maximal  $h$ -pure submodule of  $M$  such that  $\text{Soc}(K) \subseteq S$ , then by Proposition 2.2.8,  $(K + S)/K \subseteq (M/K)^1$ . Now every  $h$ -pure submodule  $A/K$  of  $M/K$  containing  $(K + S)/K$  is a summand of  $M/K$  as  $A$  is  $h$ -pure submodule of  $M$  containing  $S$ . Hence,  $M/K$  is a direct sum of a bounded submodule and a  $h$ -divisible submodule. Thus,  $M/K$  is  $h$ -pure complete, which is a contradiction. Therefore,

$Soc(K) = S$  and  $M/K$  is bounded. Hence, for some  $n$ ,  $H_n(M/K) = 0$  and we get  $Soc(H_n(M)) \subseteq S$ .

(iii)  $\rightarrow$  (ii) Let  $K$  be an  $h$ -pure submodule of  $M$  such that  $S \subseteq K$ , then  $H_n(M) \subseteq K$  and hence  $K$  is a cobounded summand of  $M$ .

**Corollary 2.2.13:** If  $M$  is a  $h$ -reduced QTAG-module and  $S$  is a subsocle of  $M$ , then every  $h$ -pure submodule  $K$  of  $M$  containing  $S$  is summand of  $M$  if and only if  $S \supseteq Soc(H_n(M))$  for some  $n$ .

**Proof:** Due to above Theorem it is sufficient to show that  $Soc(M^1) \subseteq S$ . Let  $x$  be a uniform element in  $Soc(M^1)$  and let  $x \notin S$ . Let  $K$  be a complement of  $xR$  and  $S \subseteq K$  then by Proposition 1.3.26 and Proposition 1.3.27,  $K$  is  $h$ -pure submodule of  $M$  and  $M = K \oplus D$  where  $M/K \cong D$  is  $h$ -divisible, which is a contradiction as  $M$  is  $h$ -reduced. Therefore,  $x \in S$  and we get  $Soc(M^1) \subseteq S$ .

**Proposition 2.2.14:** If  $M$  is QTAG-module and  $N$  is a submodule of  $M$  such that no proper  $h$ -pure submodule contains  $N$ . Then every  $h$ -pure submodule containing  $Soc(N)$  is a cobounded summand of  $M$ .

**Proof:** Let  $T$  be a submodule of  $M$  such that  $T \cap N = 0$ , then  $T$  is bounded, since otherwise  $T$  will contain a proper basic submodule  $B$  and we will have  $M/B = T/B \oplus K/B$ . Appealing to Proposition 1.3.13, we get  $K$  to be  $h$ -pure submodule containing  $N$ , which is a contradiction. Now let  $A$  be an  $h$ -pure submodule of  $M$  such that  $Soc(N) \subseteq A$ , then  $M/A$  has a bounded basic submodule. Otherwise, if  $B/A$  is unbounded basic submodule of  $M/A$ , then  $B = A \oplus L$  where  $L \cong B/A$  and  $A \cap N = 0$ , which is a contradiction as  $L$  is unbounded. Therefore,  $M/A = B/A \oplus D/A$  where

$B/A$  is bounded and  $D/A$  is  $h$ -divisible.

Now we show that  $D/A = 0$ . Let  $D/A \neq 0$ , then  $M/B$  is  $h$ -divisible and  $B$  is  $h$ -pure submodule of  $M$ . This implies that  $Soc(B)$  is proper dense in  $Soc(M)$  and  $Soc(N) \subseteq Soc(B)$ , which is a contradiction. Hence,  $M/A$  is bounded. As  $A$  is  $h$ -pure in  $M$ ,  $A$  is a summand of  $M$ .

**Corollary 2.2.15:** If  $M$  is QTAG-module and  $N$  is a submodule of  $M$  and  $T$  is a minimal  $h$ -pure submodule of  $M$  containing  $N$ . Then  $T = B \oplus K$  where  $B$  is bounded and  $Soc(K) = Soc(N)$ .

**Proof:** Appealing to Proposition 2.2.14 and Theorem 2.2.12, we see that  $Soc(N)$  supports an  $h$ -pure submodule  $K$  of  $T$  and  $T/K$  is bounded. Therefore,  $T = B \oplus K$ .

Let  $M$  be a QTAG-module satisfying the following:

( $\star$ )  $M/K = B/K \oplus D/K$  where  $B/K$  is bounded and  $D/K$  is  $h$ -divisible, whenever  $K$  is  $h$ -pure submodule of  $M$  containing  $M^1$ .

**Definition 2.2.16:** A QTAG-module  $M$  is called essentially finitely indecomposable (e.f.i) if it has no unbounded direct sum of uniserial submodules summand.

**Theorem 2.2.17:** If  $M$  is a QTAG-module and if  $M$  satisfies ( $\star$ ), then every  $h$ -pure submodule of  $M$  containing  $M^1$  is e.f.i..

**Proof:** Let  $A$  be an  $h$ -pure submodule of  $M$  containing  $M^1$ , then  $A$  satisfies ( $\star$ ), because if  $K$  is  $h$ -pure submodule of  $A$  containing  $A^1 = M^1$ , then  $A/K$  is  $h$ -pure submodule of  $M/K$  and the assertion follows from Proposition 2.2.9. Therefore,  $A$  satisfies ( $\star$ ). Now let  $A$  be not e.f.i., then  $A = S \oplus T$  where  $S$  is unbounded direct



sum of uniserial submodules. Therefore,  $T$  is  $h$ -pure submodule of  $A$  containing  $A^1$  and  $A/T$  is unbounded, a contradiction. Hence,  $A$  is e.f.i..

Let us consider one more condition on  $M$  introduced by S. Singh (unpublished) as mentioned below:

(A) For any finitely generated submodule  $N$  of  $M$ ,  $R/\text{ann}(N)$  is right artinian.

Now we prove the following result which is of independent interest.

**Theorem 2.2.18:** If  $M$  is a QTAG-module satisfying condition (A) and  $N$  is a quasi-essential submodule of  $M$  such that  $\text{Soc}(N) \not\subseteq M^1$ . Then every  $h$ -pure submodule  $K$  of  $M$  containing  $N$  is a cobounded summand of  $M$ .

**Proof:** Let  $K$  be  $h$ -pure submodule of  $M$  with  $N \subseteq K$ , then by Proposition 2.2.10,  $K/T$  is an absolute summand of  $M/T$  where  $T$  is any complement of  $N$  in  $K$ . Since  $\text{Soc}(N) \not\subseteq M^1$ , then Proposition 1.4.9 implies that  $K/T$  is not  $h$ -divisible for some complement  $T$  of  $N$  in  $K$ , as  $K^1 \subseteq M^1$ . Now appealing to Theorem 1.5.4, there exists a positive integer  $n$  such that

$$\text{Soc}(H_{n+1}(M/T)) \subseteq \text{Soc}(K/T) \subseteq \text{Soc}(H_n(M/T))$$

Therefore,  $\text{Soc}(H_{n+1}(M)) \subseteq K$  and as  $K$  is  $h$ -pure, then appealing to Proposition 1.3.9, we get  $H_{n+1}(M) \subseteq K$ . Hence,  $K$  is cobounded summand of  $M$ .

Now we state the following lemma, since the proof is of set theoretic nature, therefore it is omitted.

**Lemma 2.2.19:** If  $M$  is a QTAG-module such that  $M = N \oplus K$  such that  $N_0 \subseteq N$  and  $K_0 \subseteq K$  are submodules, if  $N'$  is a complement of  $N_0$  in  $N$  and  $K'$  is a comple-

ment of  $K_0$  in  $K$ , then  $N' \oplus K'$  is a complement of  $K_0 \oplus N_0$  in  $M$ .

**Proposition 2.2.20:** If  $S$  is a quasi-essential subsocle of a QATG-module  $M$  and  $N$  is an  $h$ -pure submodule of  $M$  with  $\text{Soc}(N) = \text{Soc}(H_n(M))$ . Then  $S \cap H_n(M)$  is a quasi-essential subsocle of  $N$ .

**Proof:** Let  $N_0 = S \cap H_n(M)$  and  $S = N_0 \oplus K_0$ , then trivially  $K_0 \cap H_n(M) = 0$ . Let  $K$  be a complement of  $N$  in  $M$  containing  $K_0$ ; then since  $N$  is  $h$ -pure and  $M/N$  is bounded, we get  $M = K \oplus N$ . Now let  $N'$  be a complement of  $N_0$  in  $N$  and  $T$  be an  $h$ -pure submodule of  $N$  containing  $N_0$ . If  $K'$  is complement of  $K_0$  in  $K$ , then  $N' \oplus K'$  is complement of  $S$  in  $M$  by Lemma 2.2.19. Now

$$\begin{aligned} (T \oplus K) \cap H_n(M) &= (T \oplus K) \cap (H_n(K) \oplus H_n(N)) \\ &= H_n(K) + (T \oplus K) \cap H_n(N) \end{aligned}$$

Now let  $x \in (T \oplus K) \cap H_n(N)$  then  $x = a + b, a \in T, b \in K$  and  $x \in H_n(N)$ , then  $x - a = b \in K \cap N = 0$ , so  $x \in T \cap H_n(N) = H_n(T)$ . Hence, we get

$$\begin{aligned} (T \oplus K) \cap H_n(M) &= H_n(K) \oplus H_n(T) \\ &= H_n(K \oplus T) \end{aligned}$$

So  $T \oplus K$  is an  $h$ -pure submodule of  $M$ . Trivially  $S \subseteq T \oplus K$ . Since  $S$  is quasi-essential submodule of  $M$ , we get  $M = T \oplus K + N' \oplus K' = (T + N') \oplus K$ . Hence,  $N = T + N'$ . Therefore,  $S \cap H_n(M)$  is quasi-essential in  $N$ .

**Proposition 2.2.21:** If  $S$  be a quasi-essential subsocle of a QTAG-module  $M$  satisfying condition (A) and if  $\text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M))$  for some  $n \in \mathbb{Z}^+$ , then  $S \subset \text{Soc}(H_n(M))$ .

**Proof:** Let  $A_0 = S \cap H_{n+1}(M)$  and  $S = A_0 \oplus B_0$ . Let  $\text{Soc}(H_{n+1}(M))$  support an  $h$ -pure submodule  $A$  of  $M$ . Let  $B$  be a complement of  $A$  in  $M$  such that  $B_0 \subset B$ . Then

as done in Proposition 2.2.20,  $M = A \oplus B$ . Let  $K$  be an  $h$ -pure submodule of  $B$  such that  $\text{Soc}(K) = B_0$  and  $B'$  be a complement of  $K$  in  $B$ . Then  $B'$  is also a complement of  $B_0$ . Let  $A'$  be a complement of  $A_0$  in  $A$ , then  $A' \oplus B'$  is complement of  $S$  in  $M$ . Since  $S$  is quasi-essential in  $M$  and as done in Proposition 2.2.20,  $A \oplus K$  is an  $h$ -pure submodule of  $M$  containing  $S$ . Therefore,  $M = A \oplus K + A' \oplus B' = A \oplus (K \oplus B')$ . Thus, we get  $B = K \oplus B'$ , so  $K$  is an absolute direct summand of  $B$ . Now appealing to Theorem 1.5.4, we get  $\text{Soc}(H_{k+1}(B)) \subseteq B_0 \subseteq \text{Soc}(H_k(B))$  for some  $k \in \mathbb{Z}^+$ . Since  $\text{Soc}(H_n(M)) = \text{Soc}(A) \oplus \text{Soc}(H_n(B))$  and  $\text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M))$ , we get  $\text{Soc}(H_n(B)) \subseteq B_0$ . Thus  $n \leq k$ , so  $B_0 \subseteq \text{Soc}(H_n(B))$ . Hence,  $S = A_0 + B_0 \subseteq \text{Soc}(H_{n+1}(M)) \oplus \text{Soc}(H_n(B)) = \text{Soc}(H_n(M))$ .

**Proposition 2.2.22:** If  $S$  is quasi-essential subocle of a QTAG-module  $M$  satisfying condition (A) and is  $h$ -dense in  $M$ . Then either  $S \subseteq M^1$  or  $S = \text{Soc}(M)$ .

**Proof:** Appealing to Theorem 2.2.1, we see that  $S$  supports an  $h$ -pure submodule and is quasi-essential. Now if  $S \not\subseteq M^1$ , then by Theorem 1.5.4,  $\text{Soc}(H_{k+1}(M)) \subseteq S \subseteq \text{Soc}(H_k(M))$  for some  $k \in \mathbb{Z}^+$ . Since  $\text{Soc}(M) = S + \text{Soc}(H_{k+1}(M))$  and as  $\text{Soc}(H_{k+1}(M)) \subseteq S$ , we get  $S = \text{Soc}(M)$ .

**Proposition 2.2.23:** If  $S$  be a quasi essential subocle of a QTAG-module  $M$  satisfying condition (A) and if  $\text{Soc}(H_k(M)) = (S \cap H_k(M)) + \text{Soc}(H_{k-1}(M))$  for every  $k > n$ , then either  $H_{n+1}(M)$  is  $h$ -divisible or  $\text{Soc}(H_{n+1}(M)) \subset S$ .

**Proof:** Let  $K$  be an  $h$ -pure submodule supported by  $\text{Soc}(H_{n+1}(M))$ , then  $\text{Soc}(H_k(M)) = \text{Soc}(H_k(K))$  and  $S \cap H_k(M) = S \cap H_k(K)$  for  $k > n$ , consequently  $\text{Soc}(H_k(K)) = (S \cap H_k(K)) + \text{Soc}(H_{k+1}(K))$  for every  $k > n$ . Since  $K$  is  $h$ -pure and  $\text{Soc}(H_{n+1}(M)) = \text{Soc}(K)$ , we get  $\text{Soc}(K) = \text{Soc}(H_{n+1}(K))$ . Using induction

it is easy to see that  $\text{Soc}(H_{n+1}(K)) = (S \cap H_{n+1}(K)) + \text{Soc}(H_{n+m}(K))$  for all  $m \geq 1$ . Thus  $S \cap H_{n+1}(K)$  is  $h$ -dense in  $\text{Soc}(K)$  and is quasi-essential in  $\text{Soc}(K)$  (see Proposition 2.2.20). Now by Proposition 2.2.22, either  $S \cap H_{n+1}(K) \subseteq K^1$  or  $S \cap H_{n+1}(K) = \text{Soc}(K)$ . If  $S \cap H_{n+1}(K) \subseteq K^1$ , then as  $S \cap H_{n+1}(K)$  is  $h$ -dense in  $K$ , therefore  $K$  is  $h$ -divisible; consequently  $H_{n+1}(M)$  is  $h$ -divisible. If  $S \cap \text{Soc}(H_{n+1}(K)) = \text{Soc}(K)$  then  $S \cap \text{Soc}(H_{n+1}(M)) = \text{Soc}(H_{n+1}(M))$  and we get  $\text{Soc}(H_{n+1}(M)) \subseteq S$ .

Now we state and prove the main result of this section.

**Theorem 2.2.24:** If  $M$  is a QTAG-module satisfying condition (A) and  $S$  is a subsocle of  $M$ , then  $S$  is quasi-essential if and only if one of the following conditions holds:

- (i)  $S \subseteq M^1$ .
- (ii)  $\text{Soc}(H_{n+1}(M)) \subseteq S \subseteq \text{Soc}(H_n(M))$  for some  $n \geq 0$ .

**Proof:** The sufficiency follows from Theorem 1.5.2 and Theorem 1.5.3.

Conversely, suppose  $S$  is quasi-essential. Now if  $\text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M))$  for arbitrarily large  $n$ , then by Proposition 2.2.21,  $S \subseteq M^1$ . If not so, then there exists  $n \in \mathbb{Z}^+$  such that  $\text{Soc}(H_n(M)) \neq (S \cap H_n(M)) + \text{Soc}(H_{n+1}(M))$  and equality holds for every  $k > n$ . Thus  $S \subseteq \text{Soc}(H_n(M))$  by Proposition 2.2.21 and either  $\text{Soc}(H_{n+1}(M)) \subseteq S$  or  $H_{n+1}(M)$  is  $h$ -divisible by Proposition 2.2.23. If  $\text{Soc}(H_{n+1}(M)) \subseteq S$ , then the condition (ii) is satisfied.

If  $H_{n+1}(M)$  is  $h$ -divisible then every subsocles of  $M$  will support an  $h$ -pure submodule. Thus  $S$  supports an absolute direct summand. Therefore, appealing to Theorem 1.5.4, we see that either (i) or (ii) is satisfied.

Appealing to above theorem, the following immediately follows:

**Corollary 2.2.25:** If  $M$  is a QTAG-module satisfying condition (A) then a submodule  $S$  of  $M$  supports an absolute direct summand if and only if  $S$  is quasi-essential and  $S \subset M^1$  implies  $S \subset D$ , where  $D$  is the maximal  $h$ -divisible submodule of  $M$ .

### Section-3

#### § 2.3 Minimal $h$ -pure Submodules

Firstly we recall the following definition from chapter 1.

**Definition 2.3.1:** A submodule  $N$  of a QTAG-module  $M$  is called almost dense in  $M$  if for every  $h$ -pure submodule  $K$  of  $M$  containing  $N$ ,  $M/K$  is  $h$ -divisible.

**Definition 2.3.2:** Let  $K$  be a submodule of a QTAG-module  $M$ , then an  $h$ -pure submodule  $N$  of  $M$  containing  $K$  is called minimal  $h$ -pure submodule of  $M$ .

**Theorem 2.3.3:** Let  $N$  be a submodule of a QTAG-module  $M$ . Then there is no proper  $h$ -pure submodule of  $M$  containing  $N$  if and only if  $N$  is almost dense in  $M$  and  $Soc(H_n(M)) \subseteq N$  for some  $n$ .

**Proof:** Let  $N$  be almost dense in  $M$  and  $Soc(H_n(M)) \subseteq N$ . Let  $K$  be an  $h$ -pure submodule of  $M$  such that  $N \subseteq K$ , then  $Soc(H_n(M)) \subseteq K$  and hence by Proposition 1.3.9,  $H_n(M) \subseteq K$ , consequently  $M/K$  is bounded but it is also  $h$ -divisible which is not possible and we get  $M/K = 0$  i.e.  $M = K$ .

Conversely, if no proper  $h$ -pure submodule of  $M$  contains  $N$ , clearly  $N$  is almost  $h$ -dense in  $M$  and by Theorem 2.2.12 and Proposition 2.2.14, we get  $\text{Soc}(H_n(M)) \subseteq N$  for some positive integer  $n$ .

Now we prove the following useful criterion:

**Theorem 2.3.4:** Let  $N$  be a submodule of a QTAG-module  $M$ . Then  $N$  is contained in a minimal  $h$ -pure submodule of  $M$  if and only if there exists a  $h$ -pure submodule  $K$  of  $M$  such that  $\text{Soc}(H_n(M)) \subseteq N \subseteq K$  for some  $n \in \mathbb{Z}^+$ .

**Proof:** If  $N$  is contained in a minimal  $h$ -pure submodule of  $M$  then the result follows from Theorem 1.3.29.

Conversely, suppose that there exists an  $h$ -pure submodule  $K$  of  $M$  such that  $\text{Soc}(H_n(M)) \subseteq N \subseteq K$  for some  $n \in \mathbb{Z}^+$ . If  $n = 0$ , then trivially  $K$  itself is an  $h$ -pure submodule containing  $N$ . If  $n \geq 1$ , then for every  $h$ -pure submodule  $T$  of  $K$  containing  $N$ , we define

$$E(T) = \{l \geq 1 / \text{Soc}(T_{l-1}) \not\subseteq N + H_l(T)\}$$

and set  $m(T) = 0$  if  $E(T) = \emptyset$  and  $m(T) = \max\{m \in E(T)\}$  if  $E(T) \neq \emptyset$ . Trivially,  $m(T) \leq n$  and therefore, there exists an  $h$ -pure submodule  $A$  of  $M$  containing  $N$  for which  $m(A)$  is minimal. Now by Theorem 1.3.28, we see that  $m(A) = 0$  i.e.  $A \supseteq N \supseteq \text{Soc}(H_n(A))$  and  $\text{Soc}(H_{l-1}(A)) \subseteq N + H_l(A)$  for all  $l \geq 1$ . Hence, by Theorem 1.3.29,  $A$  is a minimal  $h$ -pure submodule of  $M$  containing  $N$ .

**Theorem 2.3.5:** If  $N$  is a submodule of a QTAG-module such that  $M/N$  is a direct sum of uniserial submodules. If  $K$  is minimal  $h$ -pure submodule of  $M$  containing  $N$  then  $M/K$  is also a direct sum of uniserial submodules.

**Proof:** By Theorem 2.3.4, there exists  $n \in \mathbb{Z}^+$  such that  $\text{Soc}(H_n(K)) \subseteq N$ . Since  $K$  is  $h$ -pure in  $M$ , therefore by Proposition 1.3.10,  $\text{Soc}(H_n(M/K)) = (\text{Soc}(H_n(M)) + K)/K$ . It is trivial to see that the natural homomorphism  $f : M/N \rightarrow M/K$  defined by  $f(x + N) = x + K$  is onto and maps  $(\text{Soc}(H_n(M)) + N)/N$  onto  $(\text{Soc}(H_n(M)) + K)/K$ . Since we know that homomorphism never decreases heights. We show that  $f$  is height preserving. Let  $x$  be a uniform element in  $\text{Soc}(H_n(M))$  and  $x + K \in (\text{Soc}(H_n(M)) + K)/K$ , then we can find a uniform element  $y \in \text{Soc}(H_n(M))$  such that  $x + K = y + K$ , then trivially  $x - y \in \text{Soc}(K)$  and as  $K$  is  $h$ -pure,  $x - y \in \text{Soc}(H_n(K)) \subseteq N$ . Hence,  $x + N = y + N \in (\text{Soc}(H_n(M)) + N)/N$  and we get  $H_{M/K}(x + K) \leq H_{M/N}(x + N)$ . Since  $(\text{Soc}(H_n(M)) + N)/N$  is the union of the ascending chain of submodules of bounded height in  $M/N$ ,  $(\text{Soc}(H_n(M)) + K)/K$  is also the union of an ascending chain of submodules of bounded height in  $M/K$ . Thus,  $H_n(M/K)$  is a direct sum of uniserial submodules and  $M/K$  is direct sum of uniserial submodules.

Finally we prove the following:

**Theorem 2.3.6:** If  $N$  is a submodule of a basic submodule  $B$  of a QTAG-module  $M$ . If  $N$  is contained in a minimal  $h$ -pure submodule  $K$  of  $M$ , then  $K$  is a direct sum of uniserial submodules.

**Proof:** Since  $N \subseteq B$  and  $K$  is an  $h$ -pure submodule of  $M$ , then using Theorem 1.4.18,  $N$  can be extended to a basic submodule  $A$  of  $K$ . Since  $K$  is minimal  $h$ -pure containing  $N$ ,  $A = K$  and therefore  $K$  is direct sum of uniserial submodules.

## CHAPTER - 3

### *h*-Purifiable Submodule of QTAG-module

#### Section-1

##### § 3.1. Introduction

In general it is known that  $Soc(A + B) \neq Soc(A) + Soc(B)$ . The equality for some submodules motivated to define the concept of quasi *h*-pure submodules. The concept of quasi *h*-pure submodules were introduced by M. Z. Khan and A. Zubair and different characterizations and their consequences were obtained in [32]. In this chapter, we continue the similar study in terms of purifiability of submodules and obtained a characterization.

Section 2 deals with the purifiability of submodules. Here we define the concept of *h*-purifiable submodule of a QTAG-module and obtain the relation between purifiability of submodules and quasi *h*-pure submodules. We prove that an almost *h*-dense submodule  $N$  of a QTAG-module  $M$  is *h*-purifiable if and only if there exists  $m \in Z^+$  such that  $Q_n(M, N) = 0$  for all  $n \geq m$  (Theorem 3.2.11).

In section 3, we have discussed the role of *h*-pure and *h*-dense submodules of a submodule of a QTAG-module. Here we obtained various results which indicates that the *h*-purifiability of a submodule is very much dependent on the *h*-purifiability of *h*-pure and *h*-dense submodule of the given submodule. We have also established a necessary and sufficient condition for a submodule to be a *h*-purifiable submodule (Proposition 3.3.5). In the end of this section, we proved that a submodule  $N$  of a QTAG-module  $M$  is *h*-purifiable if and only if all the basic submodules of  $N$  are *h*-purifiable (Theorem 3.3.6).



Section 4 is devoted to the study of maximal quasi  $h$ -pure submodules of a QTAG-module. Motivated by the study of quasi  $h$ -pure submodules we have introduced the concept of maximal quasi  $h$ -pure submodules. We have also established different characterizations of quasi  $h$ -pure submodules to be maximal quasi  $h$ -pure submodules and obtained different consequences. For example; we proved that a quasi  $h$ -pure submodule  $N$  of  $M$  is maximal quasi  $h$ -pure if and only if  $N + xR$  is not quasi  $h$ -pure for every uniform element  $x \in M$  such that  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(M)$  (Theorem 3.4.12). We have also proved that if  $N$  is quasi  $h$ -pure in  $M$  then every submodule  $K$  of  $M$  such that  $N \subseteq K \subseteq \overline{N}$  is also quasi  $h$ -pure (Theorem 3.4.8). This result has got resemblance with the result of connected space in a topological space.

## Section-2

### § 3.2 Purifiability

To start with we need the following:

**Proposition 3.2.1** [Lemma 1, 27]:

- (i) For any uniform elements  $x$  and  $y \in M$  with  $x \in yR$ ,  $d(yR/xR) = m$  if and only if  $H_m(yR) = xR$ .
- (ii) If  $x$  and  $y$  are predecessors of a uniform element  $z$ , then there is an isomorphism  $\sigma : xR \rightarrow yR$  such that  $\sigma$  is identity on  $zR$ .
- (iii) For any uniform elements  $x$  and  $y \in M$ ,  $x - y \in Soc(M)$  if and only if  $H_1(xR) = H_1(yR)$ .

**Proof:** (i) is trivial and (ii) is an immediate consequence of Lemma A. For (iii), if  $x - y \in Soc(M)$ , then  $H_1(xR) = H_1(yR)$  is evident. Now let  $H_1(xR) =$

$H_1(yR) = zR$ , then  $x$  and  $y$  are predecessors of  $z$ . Hence, there is an isomorphism  $\sigma : xR \rightarrow yR$  such that  $\sigma$  is identity on  $zR$  and  $\sigma(x) = y$ . Hence, the map  $\eta : xR \rightarrow (x - \sigma(x))R$  is given as  $xr \rightarrow (x - \sigma(x))r$  is an epimorphism with  $zR \subseteq \text{Ker } \eta$ . Hence,  $e(x - y) \leq 1$  and we get  $x - y \in \text{Soc}(M)$ .

**Theorem 3.2.3 [Theorem 5, 1]:** A submodule  $N$  of a QTAG-module  $M$  is almost  $h$ -dense in  $M$  if and only if  $N + H_n(M) \supseteq \text{Soc}(H_{n-1}(M))$  for all  $n \geq 1$ .

Before defining the  $h$ -purifiable submodule, we would like to adopt the following notations and results from [32].

**Notation 3.2.4:** For any non-negative integer  $t$  and for a submodule  $N$  of a QTAG-module  $M$ , we denote by  $N^t(M)$  the submodule

$$(N + H_{t+1}(M)) \cap \text{Soc}(H_t(M))$$

and by  $N_t(M)$  the submodule

$$(N \cap \text{Soc}(H_t(M))) + \text{Soc}(H_{t+1}(M))$$

and by

$$Q_t(M, N) = N^t(M) / N_t(M)$$

**Remark 3.2.5:** It is trivial to see that

$$\begin{aligned} N^t(M) &= (N + H_{t+1}(M)) \cap \text{Soc}(H_t(M)) \\ &= \text{Soc}(N \cap H_t(M) + H_{t+1}(M)) \end{aligned}$$

and

$$\begin{aligned} N_t(M) &= (N \cap \text{Soc}(H_t(M))) + \text{Soc}(H_{t+1}(M)) \\ &= (\text{Soc}(N))^t(M) \end{aligned}$$

**Theorem 3.2.6** [Theorem 4.2, 32]: If  $N$  and  $K$  are submodules of QTAG-module  $M$  such that  $N \subseteq K$  and  $K$  is  $h$ -pure in  $M$ , then the module  $Q_n(M, N)$  and  $Q_n(K, N)$  are isomorphic, for all  $n$ .

**Proof:** Define a map

$$\sigma : N^n(K)/N_n(K) \longrightarrow N^n(M)/N_n(M)$$

such that  $\sigma(x + H_n(K)) = x + H_n(M)$ . Obviously  $\sigma$  is an  $R$ -homomorphism. Now if for some  $x \in N^n(K), x \in N_n(M)$ , then  $x = y + z$ ,  $y \in N \cap Soc(H_n(M))$  and  $z \in Soc(H_{n+1}(M))$ , then  $y \in K \cap Soc(H_n(M)) \subseteq H_n(K)$  gives  $y \in N \cap Soc(H_n(K))$ . Also  $z = x - y \in K \cap Soc(H_{n+1}(M))$  yields  $z \in Soc(H_{n+1}(K))$ . Hence  $x \in N_n(K)$  and we get  $\sigma$ , a monomorphism. We now prove that  $\sigma$  is an epimorphism. Consider  $s \in N^n(M)$  such that  $s$  is uniform and  $s \notin N_n(M)$  then  $s = a + b$ , where  $a \in N, b \in H_{n+1}(M)$ . If  $s \in N$  or  $s \in H_{n+1}(M)$ , we get  $s \in N_n(M)$ . Hence  $aR \cap sR = 0 = bR \cap sR$ . Consequently,  $aR \subseteq bR \oplus sR$  with  $a = -b + s$  gives  $aR \cong bR$  under the correspondence  $ar \leftrightarrow -br$ . Then  $H_1(aR) = H_1(bR)$  and the above correspondence is identity on  $H_1(aR)$ . Now  $a = s - b \in K \cap H_n(M) = H_n(K)$ , so that  $H_1(aR) = H_1(bR) \subseteq H_{n+2}(M) \cap K = H_{n+2}(K)$  and we get  $y \in H_{n+1}(K)$  such that  $H_1(aR) = H_1(yR)$  and  $\lambda : aR \longrightarrow yR$  given by  $\lambda(ar) = yr$  is identity on  $H_1(aR)$ . Consequently,  $e(a - y) \leq 1$ . So that  $a - y \in Soc(H_n(K))$ . Then the mapping  $\mu : bR \longrightarrow yR$  such that  $\mu(br) = -yr$  is also identity on  $H_1(bR)$  and hence  $b + y \in Soc(H_{n+1}(M))$ . Therefore,  $b + y \in N_n(M)$ . Also  $a - y \in (N + H_{n+1}(K)) \cap Soc(H_n(K))$ . Hence

$$\sigma(a - y + N_n(K)) = a - y + N_n(M) = s - (b + y) + N_n(M) = s + N_n(M).$$

This proves that  $\sigma$  is an epimorphism. Hence the result follows.

**Theorem 3.2.7 [Theorem 4.3, 32]:** If  $N$  is  $h$ -neat submodule of  $M$ , then  $N$  is  $h$ -pure in  $M$  if and only if  $Q_n(M, N) = 0$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** Let  $N$  be  $h$ -pure in  $M$  then by, Theorem 3.2.6,

$$N^t(N)/N_t(N) \equiv N^t(M)/N_t(M)$$

for all  $t > 0$ , but  $N^t(N) = N_t(N)$ . Therefore,  $N^t(M) = N_t(M)$  and we get  $Q_t(M, N) = 0$ .

Conversely, suppose  $N \cap H_n(M) = H_n(N)$ . Let  $x$  be a uniform element in  $N \cap H_{n+1}(M)$  then there is a uniform element  $y \in H_n(M)$  such that  $d(yR/xR) = 1$  and also as  $x \in N \cap H_{n+1}(M) \subseteq N \cap H_n(M) = H_n(N)$  we can find a uniform element  $z \in H_{n-1}(N)$  such that  $d(zR/xR) = 1$ . Hence  $e(y-z) \leq 1$  and so  $y-z \in Soc(M)$  but  $y-z \in N + H_n(M)$  and  $y-z \in Soc(H_{n-1}(M))$ . Therefore,  $y-z \in (N + H_n(M)) \cap Soc(H_{n-1}(M))$  but  $N^{t-1}(M) = N_{t-1}(M)$ , we get  $y-z \in N \cap Soc(H_{n-1}(M)) + Soc(H_n(M))$ . So  $y-z = a+b, a \in N \cap Soc(H_{n-1}(M)), b \in Soc(H_n(M))$ , which gives  $y-b = a+z \in N \cap H_n(M) = H_n(N)$ . Hence,

$$xR = H_1(yR) = H_1((y-b)R) \subseteq H_{n+1}(N)$$

Therefore,  $N$  is  $h$ -pure in  $M$ .

Now we are able to define  $h$ -purifiable submodule:

**Definition 3.2.8:** A submodule  $N$  of a QTAG-module  $M$  is called  $h$ -purifiable in  $M$  if there exists a submodule  $K$  of  $M$  minimal among the  $h$ -pure submodules of  $M$  containing  $N$ .

Such  $K$  is called  $h$ -pure hull of  $N$  in  $M$ .

Using this definition we restate the Theorem 2.3.4 as follows:

**Theorem 3.2.9:** A submodule  $N$  of a QTAG-module  $M$  is  $h$ -purifiable in  $M$  if and only if there exists an  $h$ -pure submodule  $K$  of  $M$  such that  $Soc(H_n(K)) \subseteq N \subseteq K$  for some  $n \in Z^+$ .

**Proposition 3.2.10:** If  $N$  is a  $h$ -purifiable submodule of a QTAG-module  $M$ ; then there exists  $m \in Z^+$  such that  $Q_n(M, N) = 0$  for all  $n \geq m$ .

**Proof:** If  $N$  itself is an  $h$ -pure submodule, then by Theorem 1.5.9,  $Q_n(M, N) = 0$  for all  $n \geq 0$ . Now appealing Theorem 3.2.9, we get an  $h$ -pure submodule  $K$  of  $M$  and  $m \in Z^+$  such that  $Soc(H_m(K)) \subseteq N \subseteq K$ . Now for  $n \geq m$  it is trivial to see that

$$\begin{aligned} N^n(K) &= (N + H_{n+1}(K)) \cap Soc(H_n(K)) \\ &= Soc(H_n(K)) \\ &= N_n(K) \end{aligned}$$

Hence,  $Q_n(K, N) = 0$  for all  $n \geq m$ . Therefore, from Theorem 3.2.6, we get  $Q_n(M, N) = 0$  for all  $n \geq m$ .

**Observation:** Using the notations used earlier, the  $h$ -purity can be established as: Since  $Soc(M) = Soc(N) + Soc(H_k(M))$  for all  $k \in Z^+$ , it is easy to see that  $N^n(M) = N_n(M)$  and  $Q_n(M, N) = 0$  for all  $n \in Z^+$ . Since  $N$  is  $h$ -neat, therefore by Theorem 1.5.9,  $N$  is  $h$ -pure in  $M$ .

**Theorem 3.2.11:** If  $N$  is almost  $h$ -dense submodule of a QTAG-module  $M$ . Then  $N$  is  $h$ -purifiable in  $M$  if and only if there exists  $m \in Z^+$  such that  $Q_n(M, N) = 0$  for all  $n \geq m$ .

**Proof:** Let  $N$  be  $h$ -purifiable then by Proposition 3.2.10, we get  $Q_n(M, N) = 0$  for all  $n \geq m$ .

Conversely, suppose that  $Q_n(M, N) = 0$  for all  $n \geq m$  and  $N$  is almost  $h$ -dense in  $M$ . Then

$$\begin{aligned} N^n(M) &= N_n(M) \\ &= \text{Soc}(N \cap H_n(M)) + \text{Soc}(H_{n+1}(M)) \end{aligned}$$

Since  $N$  is almost  $h$ -dense in  $M$ , therefore by Theorem 3.2.3, we get  $\text{Soc}(H_n(M)) = \text{Soc}(N \cap H_n(M)) + \text{Soc}(H_{n+1}(M))$  for all  $n \geq m$ . Therefore,  $\text{Soc}(N \cap H_m(M))$  is  $h$ -dense subsocle of  $H_m(M)$ . Now appealing to Theorem 2.2.1, we can find an  $h$ -pure submodule  $K$  of  $H_m(M)$  such that  $\text{Soc}(K) \subseteq N \cap H_m(M) \subseteq K$ . It is easy to see that  $H_m(M)/K$  is  $h$ -divisible submodule of  $M/K$  and  $H_m(M)/K \cap (N+K)/K = 0$ . Hence there exists a submodule  $T/K$  such that  $(N+K)/K \subseteq T/K$  and  $M/K = H_m(M)/K \oplus T/K$ . Now by Proposition 1.3.13,  $T$  is an  $h$ -pure submodule of  $M$ . Trivially  $T \cap H_m(M) = K$ , but  $T \cap H_m(M) = H_m(T)$ ; so  $H_m(T) = K$ . Hence,  $\text{Soc}(H_m(K)) \subseteq \text{Soc}(K) \subseteq N$ . Hence by Theorem 3.2.9, we get  $N$  to be  $h$ -purifiable.

### Section-3

#### § 3.3. Role of $h$ -pure and $h$ -dense Submodules

In this section we discuss the role of  $h$ -pure submodule and  $h$ -dense submodule of a given submodule on  $h$ -purifiability. We show that  $h$ -purifiability of a submodule depends upon the  $h$ -purifiability of an  $h$ -pure and  $h$ -dense submodule of the given submodule.

Firstly we prove the following results for obtaining a necessary and sufficient condition for  $h$ -purifiability.

**Theorem 3.3.1:** If  $B$  is an  $h$ -pure and  $h$ -dense submodule of a submodule  $K$  of a QTAG-module  $M$ , then  $Q_n(M, N) = Q_n(M, B)$  for all  $n \in Z^+$ .

**Proof:** Since  $B$  is  $h$ -dense in  $K$ , then we have  $K = B + H_{n+1}(K)$  for all  $n \geq 0$  and hence,  $K + H_{n+1}(M) = B + H_{n+1}(M)$ . Therefore,  $K^n(M) = B^n(M)$  for all  $n \geq 0$ . Further,

$$K_n(M) = (Soc(K))^n(M) = (Soc(K) + H_{n+1}(M)) \cap Soc(H_n(M))$$

Now appealing to Proposition 1.4.7, we get

$$\begin{aligned} K_n(M) &= \left( Soc(B) + Soc(H_{n+1}(K)) + H_{n+1}(M) \right) \cap Soc(H_n(M)) \\ &= \left( Soc(B) + H_{n+1}(M) \right) \cap Soc(H_n(M)) \\ &= B_n(M) \end{aligned}$$

Hence,  $Q_n(M, K) = Q_n(M, B)$ .

**Proposition 3.3.2:** If  $B$  is an  $h$ -pure and  $h$ -dense submodule of a submodule  $K$  of a QTAG-module  $M$ . If  $K$  is  $h$ -purifiable in  $M$ , then  $B$  is  $h$ -purifiable in  $M$ .

**Proof:** Let  $T$  be a  $h$ -pure hull of  $K$  in  $M$ . Since  $B$  is  $h$ -dense in  $K$  we get,  $K/B$  is  $h$ -divisible, so  $T/B = K/B \oplus L/B$ . Appealing to Proposition 1.3.13 we get,  $L$  to be  $h$ -pure submodule of  $T$  and hence  $L$  is  $h$ -pure in  $M$ . Let  $N$  be an  $h$ -pure submodule of  $M$  such that  $B \subseteq N \subseteq L$ . Then we claim that  $K + N$  is an  $h$ -pure submodule of  $M$ . Since  $K = B + H_n(K)$ , we have  $K + N = H_n(K) + N$ . Therefore,

$$\begin{aligned} (K + N) \cap H_n(M) &= (H_n(K) + N) \cap H_n(M) \\ &= H_n(K) + (N \cap H_n(M)) \\ &= H_n(K) + H_n(N) \\ &= H_n(K + N) \end{aligned}$$

for all  $n \geq 0$ .

Since  $T$  is a  $h$ -pure hull of  $K$  in  $M$ , we have  $K + N = T$  and

$$L = (K + N) \cap L = N + (K \cap L) = N + B = N$$

Therefore,  $L$  is a  $h$ -pure hull of  $B$  in  $M$ .

**Proposition 3.3.3:** If  $B$  is an  $h$ -pure and  $h$ -dense submodule of a submodule  $K$  of a QTAG-module  $M$  and if  $N$  be an  $h$ -pure hull of  $B$  in  $M$  and  $\text{Soc}(N) = \text{Soc}(B)$ , then  $K + N$  is an  $h$ -pure hull of  $K$  in  $M$ .

**Proof:** Since  $K/B$  is  $h$ -divisible, then appealing to Proposition 1.4.13, we have  $K = B + H_n(K)$ . Now  $K + N = B + H_n(K) + N = N + H_n(K)$  and hence

$$\begin{aligned} (K + N) \cap H_n(M) &= (N + H_n(K)) \cap H_n(M) \\ &= H_n(K) + N \cap H_n(M) \\ &= H_n(K) + H_n(N) \\ &= H_n(K + N) \end{aligned}$$

for all  $n \geq 0$ . Therefore,  $K + N$  is  $h$ -pure submodule of  $M$ . Since  $\text{Soc}(N) = \text{Soc}(B)$ , so  $\text{Soc}(K \cap N) = \text{Soc}(B)$ , so  $N \cap K$  is an essential extension of  $B$  in  $K$ . Since  $h$ -pure submodules have no proper essential extensions, therefore we get  $K \cap N = B$ . Now we show that  $\text{Soc}(K + N) = \text{Soc}(K)$ , which will yield that  $N + K$  is  $h$ -pure hull of  $K$  in  $M$ . Using Proposition 3.2.1, we can proceed as: If  $x \in \text{Soc}(K + N)$  then  $H_1(xR) = 0$  and  $x = k + t$  where  $k \in K, t \in N$ , then

$$H_1(tR) = H_1(kR) \subseteq N \cap K = B \cap H_1(K) = H_1(B).$$

Hence,  $H_1(tR) = H_1(kR) = H_1(bR)$  for  $b \in B$ . Hence,  $k - b \in \text{Soc}(K)$  and  $t + b, t - b \in \text{Soc}(N) = \text{Soc}(B)$ . Hence,  $x = k - b + b + t \in \text{Soc}(K)$  and we get  $\text{Soc}(K + N) = \text{Soc}(K)$ .



**Proposition 3.3.4:** If  $K$  is an  $h$ -pure hull of a submodule  $N$  of a QTAG-module  $M$  such that  $\text{Soc}(K) \neq \text{Soc}(N)$ . Then there exists  $m \in Z^+$  such that  $Q_m(M, N) \neq 0$ .

**Proof:** From Theorem 3.2.9 and Theorem 3.2.3, there exists  $n \in Z^+$  such that  $\text{Soc}(H_n(K)) \subseteq N$  and  $\text{Soc}(H_t(K)) \subseteq N + H_{t+1}(K)$  for all  $t \geq 0$ . Since  $\text{Soc}(K) \neq \text{Soc}(N)$ , the smallest  $n$  such that  $\text{Soc}(H_n(K)) \subseteq N$ , we have  $n \neq 0$ .

Now taking  $n = m - 1$ ,  $N^m(K) = \text{Soc}(H_m(K))$  while  $N_m(K) \subset N$ . Therefore,  $N^m(K) \neq N_m(K)$  but by Theorem 3.2.6,  $Q_m(M, N) \cong Q_m(K, N) \neq 0$ . Hence,  $Q_m(M, N) \neq 0$ .

Now we are able to prove the following necessary and sufficient condition for a submodule to be a  $h$ -purifiable submodule.

**Proposition 3.3.5:** Let  $N$  be a submodule of a QTAG-module  $M$ . if  $N$  is  $h$ -purifiable in  $M$ , then  $N \cap H_n(M)$  is  $h$ -purifiable in  $H_n(M)$  for all  $n \geq 0$ . Conversely, if  $N \cap H_n(M)$  is  $h$ -purifiable in  $H_n(M)$  for some  $n \geq 1$ , then  $N$  is  $h$ -purifiable in  $M$ .

**Proof:** Let  $K$  be  $h$ -pure hull of  $N$  in  $M$ , then trivially  $H_n(K)$  is  $h$ -pure submodule  $H_n(M)$  for all  $n \in Z^+$ . Also  $H_n(K) = K \cap H_n(M) \supseteq N \cap H_n(M)$ .

Now we claim that  $H_n(K)$  is  $h$ -pure hull of  $N \cap H_n(M)$  in  $H_n(M)$ . Let  $T$  be an  $h$ -pure submodule of  $H_n(M)$  such that  $H_n(K) \supseteq T \supseteq N \cap H_n(M)$ . Trivially  $N \cap H_n(K) \subseteq N \cap H_n(M)$  and  $N \cap H_n(K) \supseteq T \cap N \supseteq N \cap H_n(M)$ ; consequently  $H_n(K) \supseteq T \supseteq N \cap H_n(M) = N \cap H_n(K)$ . Now appealing to Theorem 1.5.10, we can extend  $N + T$  to an  $h$ -pure submodule  $D$  of  $K$  such that  $D \cap H_n(K) = T$  (we can note that  $(N + T) \cap H_n(K) = T + N \cap H_n(K) = T$ ). Thus,  $D = K$  and we get  $H_n(K) = T$ . Hence,  $H_n(K)$  is  $h$ -pure hull of  $N \cap H_n(M)$  in  $H_n(M)$ .

Conversely, let  $N \cap H_n(M)$  be  $h$ -purifiable in  $H_n(M)$  and  $T$  be  $h$ -pure hull of  $N \cap H_n(M)$  in  $H_n(M)$ . Then as done above  $(N + T) \cap H_n(M) = T$  and  $N + T$  can be extended to an  $h$ -pure submodule  $K$  of  $M$  such that  $K \cap H_n(M) = T$ . Clearly  $T = H_n(K)$ . Appealing to Theorem 3.2.9 there exists  $m \in Z^+$  such that  $\text{Soc}(H_m(T)) \subseteq H_n(M)$ ; so  $\text{Soc}(H_{m+n}(K)) \subseteq N \subseteq K$ .

Hence by Theorem 3.2.9,  $N$  is  $h$ -purifiable in  $M$ .

**Theorem 3.3.6:** If  $N$  is a submodule of a QTAG-module  $M$ . Then  $N$  is  $h$ -purifiable if and only if all basic submodules of  $N$  are  $h$ -purifiable.

**Proof:** Let all basic submodules of  $N$  be  $h$ -purifiable. Then by Theorem 3.3.1 and Theorem 3.2.11, there exists  $m \in Z^+$  such that  $Q_n(M, N) = 0$  for all  $n \geq m$ . Hence  $Q_n(H_m(M), N \cap H_m(M)) = 0$  for all  $n \geq 0$ . Let  $B$  be a basic submodule of  $N \cap H_n(M)$ ; then  $N/B = (N \cap H_n(M))/B \oplus T/B$  and we get  $T$  to be  $h$ -pure in  $N$  (see [Proposition 1.3.13]) also

$$T/B \cong N/(N \cap H_m(M)) \cong (N + H_m(M))/H_m(M)$$

is trivially bounded. Hence,  $T$  is also a direct sum of uniserial modules and we get  $T$  to be a basic submodule of  $N$ . As given,  $T$  is  $h$ -purifiable in  $M$ , therefore  $T \cap H_m(M) = B$  is  $h$ -purifiable in  $H_m(M)$  by Proposition 3.3.5; consequently  $B$  is  $h$ -purifiable basic submodule of  $N \cap H_m(M)$  in  $H_m(M)$ , and  $Q_n(H_m(M), B) = 0$  for all  $n \geq 0$ . Now let  $L$  be an  $h$ -pure hull of  $B$  in  $H_m(M)$ , then  $Q_n(L, B) = 0$  for all  $n \geq 0$ , and by Proposition 3.3.4,  $\text{Soc}(L) = \text{Soc}(B)$ .

Hence by Proposition 3.3.3,  $N \cap H_m(M)$  is  $h$ -purifiable in  $H_m(M)$  and so by Proposition 3.3.5,  $N$  is  $h$ -purifiable in  $M$ .

The converse follows from Proposition 3.3.2.

In the end of this section we prove the following result which is of particular interest.

**Theorem 3.3.7:** If  $N$  is almost  $h$ -dense submodule of a QTAG-module  $M$  and  $K$  is  $h$ -pure hull of  $Soc(N)$ . Then  $Q_n(M, N) \cong (Soc(H_n(M)) + K) / (Soc(H_{n+1}(M)) + K)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** As  $N$  is almost  $h$ -dense in  $M$ , then appealing to Theorem 3.2.3, we have  $N^n(M) = Soc(H_n(M))$ . Since  $K$  is  $h$ -pure hull of  $Soc(N)$  in  $M$ ,  $Soc(K) = Soc(N)$ . Therefore,

$$\begin{aligned} N_n(M) &= Soc(N \cap H_n(M)) + Soc(H_{n+1}(M)) \\ &= Soc(H_n(K)) + Soc(H_{n+1}(M)) \end{aligned}$$

So we get  $Q_n(M, N) = Soc(H_n(M)) / (Soc(H_n(K)) + Soc(H_{n+1}(M)))$ .

Now we define a map

$$\eta : Q_n(M, N) \longrightarrow (Soc(H_n(M)) + K) / (Soc(H_{n+1}(M)) + K)$$

given as

$$\eta(x + Soc(H_n(K)) + Soc(H_{n+1}(M))) = x + Soc(H_{n+1}(M)) + K.$$

Then trivially  $\eta$  is well defined and onto homomorphism. Now we show that  $\eta$  is one-one. Let

$$x + Soc(H_n(K)) + Soc(H_{n+1}(M)) \in Ker \eta$$

then  $x \in Soc(H_{n+1}(M)) + K$ , so  $x = y + k$ ,  $y \in Soc(H_{n+1}(M))$ ,  $k \in K$  and we get  $x - y = k \in K \cap Soc(H_n(M))$  but  $K$  is  $h$ -pure in  $M$ ; hence  $x - y \in Soc(H_n(K))$ , which yields  $x \in Soc(H_n(K)) + Soc(H_{n+1}(M))$ .

Therefore,  $Ker \eta = 0$  and we get  $\eta$  to be an isomorphism.

## Section-4

### § 3.4. Maximal Quasi $h$ -pure Submodule

In this section we introduced the concept of maximal quasi  $h$ -pure submodules. We have also given the different characterizations of quasi  $h$ -pure submodules to be maximal quasi  $h$ -pure submodules. But firstly we restate the following from [32]:

**Definition 3.4.1:** A submodule  $N$  of a QTAG-module  $M$  is called quasi  $h$ -pure if  $Q_n(M, N) = 0$  for all  $n \in Z^+$ .

**Theorem 3.4.2 [Theorem 4.8, 32]:** If  $N$  is a submodule of  $M$ , then the following hold:

- (i) If  $Soc(N)$  is  $h$ -dense in  $Soc(M)$ , then  $N$  is quasi  $h$ -pure in  $M$ .
- (ii) If  $N$  is quasi  $h$ -pure in  $M$ , then every essential submodule of  $N$  is quasi  $h$ -pure in  $M$ .

**Theorem 3.4.3 [Corollary 4.9, 32]:** If  $S$  is a  $h$ -dense subsocle of  $M$ , then any submodule  $N$  with  $Soc(N) \subseteq S$  can be extended to an  $h$ -pure submodule  $K$  of  $M$  such that  $Soc(K) = S$ .

**Proposition 3.4.4 [Proposition 4.10, 32]:** If  $N$  is a submodule of  $M$ , then the following hold:

- (i)  $Q_{m+n}(M, N) = Q_m(H_n(M), N \cap H_n(M))$  for all  $n, m \geq 0$ .
- (ii)  $Q_j(M, N) = 0$  for  $j = 0, 1, \dots, n$  if and only if  $Soc(N + H_t(M)) = Soc(N) + Soc(H_t(M))$  for  $t = 1, 2, \dots, n + 1$ .

**Theorem 3.4.5 [Proposition 4.14, 32]:** If  $N$  is quasi  $h$ -pure in  $M$  and  $Soc(N) \subseteq \cap_1^\infty H_n(M)$ , then  $N \subseteq \cap_1^\infty H_n(M)$ .

**Proof:** Suppose every uniform element of  $N$  of exponent  $t$  lies inside  $\cap H_n(M)$ . Let  $x$  be a uniform element in  $N$  such that  $e(x) = t + 1$ . Then we can find a uniform element  $y \in xR$  such that  $d(xR/yR) = 1$ . Hence  $y \in \cap H_n(M)$  and we get  $y \in H_n(M)$  for every  $n$ . Consequently, there is a uniform element  $z_i \in H_i(M)$  such that  $d(z_i R/yR) = 1$  which in turn will give  $e(x - z_i) \leq 1$ . So  $x - z_i \in Soc(N + H_i(M)) = Soc(N) + Soc(H_i(M))$ . Let  $x - z_i = u + v$ ,  $u \in Soc(N)$  and  $v \in Soc(H_i(M))$ . Since  $Soc(N) \subset \cap H_n(M)$ , so  $x \in \cap H_n(M)$  and we get  $N \subseteq \cap_1^\infty H_n(M)$ .

**Definition 3.4.6 [4]:** The closure of a submodule  $N$  of a QTAG-module  $M$ , denoted by  $\overline{N}$  is defined as  $\overline{N}/N = (M/N)^1$ .  $N$  is called closed if  $\overline{N} = N$ .

It is trivial to see that  $\overline{N} = \cap_0^\infty (N + H_n(M))$ .

Now we are able to define maximal quasi  $h$ -pure submodule.

**Definition 3.4.7:** A submodule  $N$  of a QTAG-module  $M$  is called maximal quasi  $h$ -pure if it is maximal among all its quasi  $h$ -pure submodules supported by  $Soc(N)$ .

Firstly we prove that the closure of a quasi  $h$ -pure submodule is quasi  $h$ -pure.

**Theorem 3.4.8:** If  $N$  is quasi  $h$ -pure submodule of a QTAG-module  $M$ . Then every submodule  $K$  of  $M$  such that  $N \subseteq K \subseteq \overline{N}$  is quasi  $h$ -pure.

**Proof:** Since  $\overline{N} = \cap_{n=0}^{\infty} (N + H_n(M))$ , we get  $K \subseteq N + H_{n+1}(M)$ . Now

$$\begin{aligned} N^n(M) &= (N + H_{n+1}(M)) \cap \text{Soc}(H_n(M)) \\ &= (K + H_{n+1}(M)) \cap \text{Soc}(H_n(M)) \\ &= K^n(M) \end{aligned}$$

Now define a map  $\sigma : Q_n(M, N) \longrightarrow Q_n(M, K)$  such that  $\sigma(x + N_n(M)) = x + K_n(M)$ ; then trivially  $\sigma$  is well defined epimorphism.

Hence,  $\sigma(Q_n(M, N)) = Q_n(M, K)$ . Now if  $N$  is quasi  $h$ -pure then  $Q_n(M, N) = 0$  and so  $Q_n(M, K) = 0$ . Therefore,  $K$  is quasi  $h$ -pure and in particular  $\overline{N}$  is also quasi  $h$ -pure.

**Proposition 3.4.9:** Every subsocle of a QTAG-module  $M$  supports a maximal quasi  $h$ -pure submodule.

**Proof:** Let  $S \subseteq \text{Soc}(M)$ , then by Proposition 1.5.7,  $S$  is quasi  $h$ -pure in  $M$ . Let  $F = \{N \subseteq M \text{ (a submodule)} \mid \text{Soc}(N) = S \text{ and } N \text{ is quasi } h\text{-pure in } M\}$ . Trivially  $F \neq \emptyset$  as  $S \in F$ . The existence of maximal element is ensured by Zorn's lemma and then appealing to Theorem 1.5.8, we get a maximal quasi  $h$ -pure submodule supported by  $S$ .

Now, appealing to Theorem 3.4.5, we have the following:

**Corollary 3.4.10:** If  $M$  is a QTAG-module then  $M^1$  is the unique maximal quasi  $h$ -pure submodule of  $M$  supported by  $\text{Soc}(M^1)$ .

**Proposition 3.4.11:** If  $N$  is a maximal quasi  $h$ -pure submodule of  $M$  such that for any uniform element  $x \in M$ ,  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(N)$ , then

$H_{M/N}(x + N) = m$  and  $Q_m(M, (N + xR)) \neq 0$ , while  $Q_n(M, (N + xR)) = 0$  for all  $n \neq m$ .

**Proof:** It is clear to see that due to  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(N)$ ,  $\text{Soc}(N + xR) = \text{Soc}(N)$  and  $x \notin N$ . Since  $N$  is maximal quasi  $h$ -pure,  $(N + xR)$  is not quasi  $h$ -pure and also appealing to Theorem 3.4.8 we get  $x \notin \overline{N}$ . Therefore,  $H_{M/N}(x + N) = m < \infty$ . Now we show that  $Q_n(M, (N + xR)) = 0$  for all  $n \neq m$ . On contrary suppose  $Q_n(M, (N + xR)) \neq 0$ , then there exists a uniform element  $z \in (N + xR)^n(M)$  such that  $z \notin (N + xR)_n(M)$ . In other words,

$$z \notin (N + xR) \cap \text{Soc}(H_n(M)) + \text{Soc}(H_{n+1}(M)) = N \cap \text{Soc}(H_n(M)) + \text{Soc}(H_{n+1}(M)).$$

As  $z \in (N + xR)^n(M) = ((N + xR) + H_{n+1}(M)) \cap \text{Soc}(H_n(M))$ , we get

$$z = t + xr + s \in \text{Soc}(H_n(M))$$

where  $t \in N, r \in R, s \in H_{n+1}(M)$ . Now either  $xrR = xR$  or  $xrR = xR \cap N$ . If  $xrR = xR \cap N$  then  $z \in N^n(M) = N_n(M) \subseteq (N + xR)_n(M)$ , a contradiction. Therefore,  $xrR = xR$  and without any loss of generality we may suppose  $z = t + x + s$ , and get  $t + x \in H_n(M)$  and so  $x + N \in H_n(M/N)$ , therefore  $n \leq m$  as  $H_{M/N}(x + N) = m$ . Hence,  $Q_n(M, (N + xR)) = 0$  for all  $n > m$ . Now suppose  $n < m$ . Now since  $x + N \in H_m(M/N)$ , there exists an element  $u \in N$  such that  $x + u \in H_m(M) \subseteq H_{n+1}(M)$ . Now we have

$$z = (t - u) + (u + x) + s \in N^n(M) = N_n(M),$$

which is a contradiction. Therefore,  $Q_n(M, (N + xR)) = 0$  for all  $n \neq m$ . Hence,  $Q_m(M, (N + xR)) \neq 0$  for otherwise  $N + xR$  will be quasi  $h$ -pure.

**Theorem 3.4.12:** If  $N$  is a quasi  $h$ -pure submodule of a QTAG-module  $M$ . Then  $N$  is maximal quasi  $h$ -pure if and only if  $N + xR$  is not quasi  $h$ -pure for every uni-

form element  $x \in M$  such that  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(M)$ .

**Proof:** Let  $N$  be maximal quasi  $h$ -pure, then by Proposition 3.4.11,  $N + xR$  is not quasi  $h$ -pure for every uniform element  $x \in M$  such that  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(M)$ .

Conversely, suppose that  $N + xR$  is not quasi  $h$ -pure such that  $d(xR/(xR \cap N)) = 1$  and  $xR \cap N \not\subseteq H_1(M)$  for any uniform element  $x \in M$ . Now let  $K$  be a quasi  $h$ -pure submodule of  $M$  such that  $\text{Soc}(K) = \text{Soc}(N)$ . Now we show that  $K = N$ . Let  $x$  be a uniform element in  $K$  such that  $d(xR/(xR \cap N)) = 1$ . Let  $xR \cap N = yR$ , then  $y \in H_1(N)$  because if  $y \notin H_1(N)$  i.e.  $yR = xR \cap N \not\subseteq H_1(N)$  then  $N + xR$  can not be quasi  $h$ -pure but by Theorem 3.4.2 (ii) every submodule of  $K$  containing  $\text{Soc}(N)$  is quasi  $h$ -pure. Therefore,  $y \in H_1(N)$ , so we can get a uniform element  $z \in N$  such that  $d(zR/yR) = 1$ . Therefore,  $e(x - z) \leq 1$ , so  $x - z \in \text{Soc}(K) = \text{Soc}(N)$ . Hence, we get  $x \in N$ , consequently  $K = N$  and we get the maximality of  $N$ .

**Proposition 3.4.13:** If  $S$  is a  $h$ -dense subsocle of a QTAG-module  $M$ . Then the maximal quasi  $h$ -pure submodules of  $M$  supported by  $S$  are  $h$ -pure.

**Proof:** Appealing to Theorem 3.4.2 (i) and Theorem 3.4.3 we get the result.

**Proposition 3.4.14:** If  $N$  is maximal quasi  $h$ -pure submodule of a QTAG-module  $M$ . Then the following hold:

- (i)  $N$  is  $h$ -neat in  $\overline{N}$ .
- (ii) If  $\overline{N}$  is  $h$ -pure in  $M$  then  $N$  is also  $h$ -pure in  $M$ .

**Proof:** (i) Appealing to Theorem 3.4.8, we get the result.



(ii) Since  $N$  is  $h$ -neat in  $\overline{N}$  and  $\overline{N}$  is  $h$ -pure in  $M$ , therefore  $N$  is  $h$ -neat in  $M$  and  $N$  being quasi  $h$ -pure and we get  $N$  to be  $h$ -pure in  $M$  [see (Theorem 3.2.7)].

**Theorem 3.4.15:** If  $N$  is a maximal quasi  $h$ -pure submodule of a QTAG-module  $M$ . Then  $N \cap H_n(M)$  is maximal quasi  $h$ -pure submodule of  $H_n(M)$ .

**Proof:** It suffices to prove that  $N \cap H_1(M)$  is maximal quasi  $h$ -pure submodule of  $H_1(M)$ . Let  $x$  be a uniform element in  $H_1(M)$  such that  $d\left(xR/(xR \cap N \cap H_1(M))\right) = 1$  and  $xR \cap (N \cap H_1(M)) \not\subseteq H_1(N \cap H_1(M))$ . Now let  $yR = xR \cap N \cap H_1(M)$  then  $y \in N$ , but  $y \notin H_1(N)$  for otherwise  $xR \cap N \cap H_1(M) \subseteq H_1(N \cap H_1(M))$  as by Theorem 1.5.8,  $H_1(N \cap H_1(M)) = H_1(N) \cap H_2(M)$ . Therefore,  $\text{Soc}(N + xR) = \text{Soc}(N)$  and  $x \notin N$ . Hence,  $N + xR$  is not quasi  $h$ -pure in  $M$ . Trivially  $H(x + N) \geq 1$ , as  $x \in H_1(M)$ . Appealing to Proposition 3.4.11, we get  $H(x + N) = m \geq 1$  and  $Q_m(M, (N + xR)) \neq 0$ . Now appealing to Proposition 3.4.4 (i), we get

$$Q_m(M, (N + xR)) = Q_{m-1}(H_1(M), (N + xR) \cap H_1(M)).$$

As  $xR \subseteq H_1(M)$ , we have  $Q_{m-1}(H_1(M), (xR + N \cap H_1(M))) = Q_m(M, (N + xR)) \neq 0$ . Hence,  $xR + N \cap H_1(M)$  is not quasi  $h$ -pure in  $H_1(M)$ . Therefore, by Theorem 3.4.12,  $N \cap H_1(M)$  is maximal quasi  $h$ -pure in  $M$ .

Repeating the same arguments  $n$  times we get  $N \cap H_n(M)$  to be maximal quasi  $h$ -pure submodule of  $H_n(M)$ .

# CHAPTER - 4

## CENTER OF $h$ -PURITY IN QTAG-MODULES

### Section-1

#### § 4.1. Introduction

The concept of center of  $h$ -purity in QTAG-modules was defined by Khan [18]. The purpose of this chapter is essentially to study center of  $h$ -purity and their characterizations. We have further studied subsocles of QTAG-modules and their interesting properties about height and range establishing various facts about the same.

Section 2 is devoted to the study of center of  $h$ -purity. It has been seen in [22] that every submodule of  $M^1$  is center of  $h$ -purity and for any  $k \geq 1$ ,  $H_k(M)$  is center of  $h$ -purity in  $M$ . In this section we generalize some of results of Ried [41] and Pierce [40] for QTAG-modules.

Section 3 deals with the study of subsocles and their some properties about height and range. Here we also define a term open subsocle  $S$  of a QTAG-module  $M$ . We have proved that if  $S$  is a open subsocle of  $M$  with height  $k$ , then  $\text{range}(S) \leq n + 1$  if and only if  $\text{range}(\text{Soc}(M)/T) \leq n$  (Theorem 4.3.6).

In section 4, we define a new concept of  $n$ - $h$ -purity, where  $n$  is a non-negative integer. The concept of  $n$ - $h$ -purity generalizes the concept of  $h$ -purity. It is evident that if  $n = 0$  then  $n$ - $h$ -purity is simply  $h$ -purity. We have established that a subsocle  $S$  of a QTAG-module  $M$  becomes center of  $n$ - $h$ -purity if and only if either  $h(S) = \infty$  or  $S$  is open such that  $\text{range}(S) \leq n + 2$  (Theorem 4.4.6).

In section 5, we discuss about a special type of QTAG-module obtained by laying down some restrictions on heights of elements of QTAG-module and some characterizations in this regard has been obtained. We have established that such restrictions on a QTAG-module make the module either  $h$ -divisible or decomposable.

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## Section-2

### § 4.2. Center of $h$ -purity

Before we start, we recall the following definition from [18].

**Definition 4.2.1:** Let  $M$  be a QTAG-module and  $N$  be a submodule of  $M$  then  $N$  is called center of  $h$ -purity in  $M$  if every complement of  $N$  in  $M$  is  $h$ -pure submodule of  $M$ .

[Theorem 7, 22] shows that every submodule of  $M^1$  is center of  $h$ -purity. Also [Corollary 10, 22] shows that for any  $k \geq 1$ ,  $H_k(M)$  is center of  $h$ -purity in  $M$ .

Now using the similar technique of Lemma 3.2.1 we can easily prove the following:

**Proposition 4.2.2:** If  $M$  is a QTAG-module and  $x, y$  are uniform elements in  $M$  then following hold:

- (i)  $x - y \in Soc^n(M)$  if and only if  $H_n(xR) = H_n(yR)$ .

(ii) For every element  $t \in \text{Soc}(M)$ ,  $H_1((x+t)R) = H_1(xR)$ .

Now we prove the following theorem which generalizes [Theorem 2.1, 41].

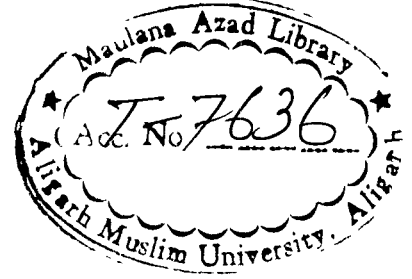
**Theorem 4.2.3 :** If  $M$  is a QTAG-module and  $N$  is a submodule of  $M$ . Then there exists a submodule  $K$  of  $M$  such that  $K$  is maximal with respect to  $K \cap N = 0$  and  $K$  is not  $h$ -pure in  $M$  if and only if the following condition is satisfied:

( $\star$ ) there exists uniform element  $u \in N$  and  $v \in M$  such that  $u + v$  is uniform and

(i)  $e(v) > e(u) = 1$

(ii)  $H(v) = H(u) < H(u + v)$

(iii)  $vR \cap N = 0$



**Proof :** Let  $K$  be a submodule of  $M$ , maximal with respect to  $K \cap N = 0$  and  $K$  be not  $h$ -pure in  $M$ . Let  $n$  be the least positive integer such that  $K \cap H_n(M) \neq H_n(K)$ , then appealing to Proposition 1.3.21, we have  $n \geq 2$ . Let  $x$  be a uniform element in  $K \cap H_n(M)$ , then there exists a uniform element  $y \in M$  such that  $y \notin K$ ,  $x \in yR$  and  $d(yR/xR) = n$ . Let  $zR/xR$  be a submodule of  $yR/xR$  such that  $d(zR/xR) = 1$ , then  $d(yR/zR) = n - 1$ . By  $h$ -neatness of  $K$ , there exists a uniform element  $t \in K$  such that  $x \in tR$  and  $d(tR/xR) = 1$ .

Hence, there exists an isomorphism  $\sigma : zR \rightarrow tR$  which is the identity on  $xR$ . Trivially  $e(z - \sigma(z)) \leq 1$ , so  $z - \sigma(z) = u + w$  where  $u \in \text{Soc}(N)$  and  $w \in \text{Soc}(K)$ . It is easy to see that  $u$  and  $w$  are uniform. Let  $H(u) \geq n - 1$ , then we can find a uniform element  $s \in M$  such that  $d(sR/uR) = n - 1$ . Now  $z - u = w + \sigma(z) \in K$  and  $z - u \in H_{n-1}(M)$ , so  $z - u = w + \sigma(z) \in K \cap H_{n-1}(M) = H_{n-1}(K)$ . Since  $(w + \sigma(z))R$  is homomorphic image of  $zR$ ,  $w + \sigma(z)$  is an uniform element.

Now we can find a uniform element  $w' \in K$  such that  $w + \sigma(z) \in w'R$  and

$d(w'R/(w + \sigma(z))R) = n - 1$ . Trivially  $d(w + \sigma(z))R > 1$ , so we can find a submodule  $gR \subseteq (w + \sigma(z))R$  such that  $d((w + \sigma(z))R/gR) = 1$ . Now appealing to Proposition 3.2.1 and 4.2.2 we get,  $H_1(zR) = xR$  and

$$\begin{aligned} H_1((w + \sigma(z))R) &= gR \\ &= H_1(\sigma(z)R) \\ &= H_1(zR) \\ &= xR \end{aligned}$$

which in turn gives  $x \in H_n(K)$ , a contradiction. Hence  $H(u) < n - 1$ . Let  $v = w + \sigma(z)$  then  $e(v) > e(u) = 1$  and  $H(u) = H(v) < H(z) = H(u + v)$ , since  $v \in K$ ,  $vR \cap N = 0$ .

Therefore the conditions of the theorem are satisfied.

Conversely, suppose that the conditions are satisfied. Let for some natural number  $n$ ,  $H(v) < n \leq H(u + v)$  and  $T_n = \text{Soc}(H_n(M))$ . Since  $e(v) > e(u) = 1$ ,  $e(v) \geq 2$ . Let  $zR = \text{Soc}(vR)$ , then  $d(vR/zR) \geq 1$  and we get  $zR \subseteq H_1(vR)$ . Also  $H_1((u + v)R) = H_1(vR) \supseteq zR$ , consequently  $z \in T_n$ . Since  $vR \cap N = 0$  and  $z \notin N$ . Let  $T_n = S \oplus T_n \cap \text{Soc}(N)$  and  $z \in S$ . Also (ii) gives  $u \notin T_n \cap \text{Soc}(N)$ , so  $\text{Soc}(N) = T \oplus (T_n \cap \text{Soc}(N))$ ,  $u \in T$ . Now  $T_n + \text{Soc}(N) = S \oplus T \oplus (T_n \cap \text{Soc}(N))$ . Similarly, we get  $\text{Soc}(M) = L \oplus (T_n + \text{Soc}(N))$  for some subsocle  $L$ . Let  $T_0 = L \oplus S$  then  $\text{Soc}(M) = T_0 \oplus \text{Soc}(N)$ , with  $z \in T_0$ .

Let  $\pi$  be the projection of  $\text{Soc}(M)$  onto  $\text{Soc}(N)$  then  $\pi(T_n) = (T_n \cap \text{Soc}(N))$ . Let  $U = T_0 + vR$ , then

$$\begin{aligned} \text{Soc}(U) &= T_0 + \text{Soc}(vR) \\ &= T_0 + zR \\ &= T_0 \end{aligned}$$

Therefore,  $\text{Soc}(U) \cap \text{Soc}(N) = 0$  and we get  $U \cap N = 0$ . Now we embed  $U$  into a complement  $K$  of  $N$ . Let  $tR$  be a submodule of  $vR$  such that  $d(vR/tR) = 1$ . As

$H_1((v+u)R) = H_1(vR) = tR$ , we get  $H(t) \geq n+1$ .

Now we show that  $H_K(t) \leq n$ . Let  $H_K(t) \geq n+1$  then there exists a uniform element  $y \in K$  such that  $t \in yR$  and  $d(yR/tR) = n+1$ . Let  $wR/tR$  be a submodule of  $yR/tR$  such that  $d(wR/tR) = 1$  and  $d(yR/wR) = n$ .

Hence, there exists an isomorphism  $\sigma : vR \rightarrow wR$  which is the identity on  $tR$ . The map  $\eta : vR \rightarrow (v - \sigma(v))R$  is an  $R$ -epimorphism with  $tR \leq \text{Ker } \eta$ . Hence,  $e(v - \sigma(v)) \leq 1$  and we get  $v - \sigma(v) \in \text{Soc}(M)$ . Since,  $H(u+v) \geq n$ ,  $u+v \in H_n(M)$ .

Therefore,  $u + v - \sigma(v) \in H_n(M)$ , consequently

$$u + v - \sigma(v) \in \text{Soc}(M) \cap H_n(M) = T_n.$$

Also  $v - \sigma(v) \in K$ , so  $v - \sigma(v) \in K \cap \text{Soc}(M) = K \cap (T_0 + \text{Soc}(N)) = T_0$ .

Therefore,

$$u = \pi(u + v - \sigma(v)) \in \pi(T_n) = T_n \cap \text{Soc}(N)$$

and we get  $H(u) \geq n$  but  $H(u) = H(v) < n$ .

Hence, we reach at a contradiction. This shows that  $H_K(t) \leq n$ . Therefore,  $K$  is not  $h$ -pure in  $M$ .

Using the above theorem we prove the following generalization of [Theorem 1, 40]. It may be noticed that the proof given below has similarity with the corresponding proof in [Theorem 1, 40].

**Theorem 4.2.4:** Let  $M$  be a QTAG-module and  $T_n = \text{Soc}(H_n(M))$ ,  $T_\infty = \text{Soc}(M^1)$  and  $T_{\infty+1} = T_{\infty+2} = 0$ . Let  $N$  be a submodule of  $M$  then  $N$  is center of  $h$ -purity in  $M$ , if and only if there exists  $k$  with  $0 \leq k \leq \infty$  such that  $T_k \supseteq \text{Soc}(N) \supseteq T_{k+2}$ .

**Proof:** Let for some  $n$ ,  $T_n \supseteq \text{Soc}(N) \supseteq T_{n+2}$ . Suppose  $N$  is not center of  $h$ -purity in  $M$ . Now if  $n = \infty$  then there does not exist any uniform element in  $\text{Soc}(N)$

satisfying condition (ii) of Theorem 4.2.3. Suppose  $n$  is finite.

Let  $u \in \text{Soc}(N)$ ,  $v \in M$  be uniform elements satisfying conditions of Theorem 4.2.3. Let  $H(u) = k$  then as  $u \in T_n$ ,  $n \leq k < H(u + v)$ . Since  $e(v) > e(u) = 1$ , we can find a submodule  $tR$  of  $vR$  such that  $d(vR/tR) = 1$ . Let  $w = u + v$  then

$$\begin{aligned} H_1((u + v)R) &= H_1(vR) \\ &= tR \end{aligned}$$

Let  $zR = \text{Soc}(vR)$  then as  $vR$  is totally ordered  $zR \leq tR$ . Hence  $H(z) \geq n + 2$ . This shows that  $z \in T_{n+2} \supseteq \text{Soc}(N)$  and we get a contradiction to the fact that  $vR \cap N = 0$ . Therefore,  $N$  is center of  $h$ -purity in  $M$ .

Conversely, suppose  $T_n \supseteq \text{Soc}(N) \supseteq T_{n+2}$  is not true for any  $n$ . Then  $\text{Soc}(N) \not\subseteq M^1$ , so  $\text{Soc}(N) \not\subseteq T_m$  for some  $m$ . Let  $k$  be the greatest natural number such that  $\text{Soc}(N) \subset T_k$ . Then the maximality of  $k$  and the assumption yield  $\text{Soc}(N) \not\subseteq T_{k+1}$  and  $T_{k+2} \not\subseteq \text{Soc}(N)$ . Hence, there exist uniform elements  $u \in \text{Soc}(N)$  and  $s \in T_{k+2}$  such that  $H(u) = k$  and  $s \notin \text{Soc}(N)$ .

Now we can find a uniform element  $y \in M$  such that  $s \in yR$  and  $d(yR/sR) = k + 2$ . Let  $xR/sR$  be a submodule of  $yR/sR$  such that  $d(xR/sR) = 1$ , then  $d(yR/xR) = k + 1$ ,  $e(x) = 2$  and we get  $H(x) \geq k + 1$ . Let  $v = x - u$ , then

$$\begin{aligned} H_1((x - u)R) &= H_1(vR) \\ &= H_1(xR) \\ &= sR \end{aligned}$$

Consequently,  $s \in (x - u)R$ . Hence,  $s = (x - u)r$  for some  $r \in R$ . If  $xr = 0$ , then  $ur = 0$  otherwise,  $s \in \text{Soc}(N)$ .

Define a map  $\eta : xR \rightarrow (x - u)R$  given as  $xr \rightarrow (x - u)r$  then  $\eta$  is a well defined onto homomorphism, consequently  $v = x - u$  is a uniform element. Trivially  $H(v) = k$  and  $H(u + v) = H(x) \geq k + 1$ . Since  $e(x) = 2$  and  $e(u) = 1$ ,  $e(v) = 2 > e(u)$ . Now suppose  $vR \cap N \neq 0$  then there exists a uniform element  $x' \in vR \cap N$  and  $x' = vr$

for some  $r \in R$ . Now  $x' = vr = xr - ur$ . Trivially  $xr \neq 0$ , so either  $xrR = xR$  or  $xrR = sR$  and in each case we get  $s \in N$  which is a contradiction. Therefore,  $vr \cap N = 0$ . Hence, by Theorem 4.2.3,  $N$  is not a center of  $h$ -purity in  $M$ . This completes the proof of the theorem.

### Section-3

#### § 4.3. Height of Subsocles

In this section we talk about subsocle and their some properties about height and range. We introduce here open subsocles of QTAG-module. Firstly we give the following definitions:

**Definition 4.3.1:** Let  $S$  be a subsocle of a QTAG-module  $M$ , then height of  $S$  is defined as a non-negative integer  $k$  such that  $S \subseteq H_k(M)$  but  $S \not\subseteq H_{k+1}(M)$  and we write  $h(S) = k$ .

If no such  $k$  is possible then we write  $h(S) = \infty$ , so  $S \subseteq M^1$ .

**Definition 4.3.2:** A subsocle  $S$  of a QTAG-module  $M$  is called open if  $Soc(H_n(M)) \subseteq S$  for some non-negative integer  $n$ .

**Definition 4.3.3:** If  $S$  is open subsocle of a QTAG-module  $M$  with  $h(S) = k$  then the range of  $S$  is the least non-negative integer  $n$  such that  $Soc(H_{k+n}(M)) \subseteq S$  and we write  $range(S) = n$ .

Now from Theorem 4.2.4, it is evident that a subsocle  $S$  of finite height is center of  $h$ -purity if and only if  $range(S) \leq 2$ .



**Proposition 4.3.4:** Let  $S$  be a subsocle of a QTAG-module  $M$  and  $n$  be any non-negative integer then

$$(1) \ S \cap H_{n+1}(M) = 0 \text{ if and only if } Soc(H_n(M/S)) \subseteq Soc(M)/S.$$

$$(2) \ S + Soc(H_n(M)) = Soc(M) \text{ if and only if } Soc(M)/S \subseteq H_n(M/S).$$

**Proof:** (1) Let  $S \cap H_{n+1}(M) = 0$  and  $\bar{x} \in Soc(H_n(M/S)) = Soc((H_n(M) + S)/S)$ , then  $x \in H_n(M)$  and  $H_1(\bar{x}R) = 0$  which in turn implies  $H_1(xR) \subseteq S$ , so

$$H_1(xR) \subseteq S \cap H_{n+1}(M) = 0$$

Therefore,  $x \in soc(M)$  and we get  $Soc(H_n(M/S)) \subseteq Soc(M)/S$ .

Conversely, suppose  $S \cap H_{n+1}(M) \neq 0$ . Let  $x$  be a uniform element in  $S \cap H_{n+1}(M)$ , then there is a uniform element  $y \in M$  such that  $d(yR/xR) = n+1$ . Let  $zR/xR = Soc(yR/xR)$ , then  $d(yR/zR) = n$  and  $d(zR/xR) = 1$ , so  $z \in H_n(M)$  and  $H_1(zR) = xR \subseteq S$ . Now  $H_1(\bar{z}R) = \bar{0}$ , so we get  $\bar{z} \in Soc(H_n(M/S)) \subseteq Soc(M)/S$ , which gives  $z \in Soc(M)$  but this is not possible. Therefore  $S \cap H_{n+1}(M) = 0$ .

(2) Let  $Soc(M) = S + Soc(H_n(M))$  and  $\bar{x} \in Soc(M)/S$ , then  $\bar{x} = y + S$ , where  $y \in Soc(H_n(M))$ , consequently  $\bar{x} \in H_n(M/S)$ .

Conversely, if we take  $x \in Soc(M)$  then  $x + S = z + S$  where  $z \in H_n(M)$ . Hence  $x = z + s$ ,  $s \in S$  and we get  $Soc(M) = S + Soc(H_n(M))$ .

**Proposition 4.3.5:** Let  $S$  be a subsocle of a QTAG-module  $M$  such that  $h(S) = k$  and  $Soc(H_{k+n+1}(M)) \not\subseteq S$  for some integer  $n \geq 0$ . Then there exists a complementary subsocle  $T$  of  $S$  in  $Soc(M)$  such that  $h(Soc(M)/T) = k$  and  $Soc(H_{k+n}(M/T)) \not\subseteq$

$Soc(M)/T$ .

**Proof:** Trivially  $S \cap Soc(H_{k+n+1}(M)) \subsetneq Soc(H_{k+n+1}(M))$ . Since  $Soc(H_{k+n+1}(M))$  is bounded, we shall have

$$Soc(H_{k+n+1}(M)) = T_0 \oplus S \cap Soc(H_{k+n+1}(M)).$$

It is easy to see that  $T_0 \cap S = 0$  and  $T_0 \subseteq H_{k+1}(M)$ . As  $S \cap H_{k+1}(M) \oplus T_0 \subseteq Soc(H_{k+1}(M))$ , we can find a subsocle  $T_1$  such that

$$Soc(H_{k+1}(M)) = S \cap H_{k+1}(M) \oplus T_0 \oplus T_1.$$

Now using the definition of height of  $S$ , we will have  $S \cap H_{k+1}(M) \subsetneq S$ .

Hence,  $S = S \cap H_{k+1}(M) \oplus S'$  for some subsocle  $S'$ . Trivially  $S' \subseteq H_k(M)$  and  $S' \cap H_{k+1}(M) = 0$ . Since  $Soc(H_{k+1}(M)) \oplus S' \subseteq Soc(H_k(M))$ , we get a subsocle  $T_2$  such that  $Soc(H_k(M)) = Soc(H_{k+1}(M)) \oplus S' \oplus T_2$ . Trivially  $S \cap (T_0 \oplus T_1 \oplus T_2) = 0$ . Let  $Soc(M) = Soc(H_k(M)) \oplus T_3$  and  $T = T_0 \oplus T_1 \oplus T_2 \oplus T_3$  then

$$\begin{aligned} Soc(M) &= Soc(H_k(M)) \oplus T_3 \\ &= Soc(H_{k+1}(M)) + S' + T_2 + T_3 \\ &= S \cap Soc(H_{k+1}(M)) \oplus T_0 \oplus T_1 \oplus S' \oplus T_2 \oplus T_3 \\ &= S \oplus T \end{aligned}$$

Hence,  $(S+T)/T = Soc(M)/T \subseteq H_k(M)/T$ . Now since  $T_0 \neq 0$ ,  $T \cap H_{k+n+1}(M) \neq 0$  and consequently, by Proposition 4.3.4,  $Soc(H_{k+n}(M/T)) \not\subseteq Soc(M)/T$ . Also as  $Soc(M) \neq T + Soc(H_{k+1}(M))$ , appealing to Proposition 4.3.4, we get  $Soc(M)/T \not\subseteq H_{k+1}(M/T)$ . Hence  $h(Soc(M)/T) = k$ .

**Theorem 4.3.6:** Let  $S$  be a open subsocle of a QTAG-module  $M$  such that  $h(S) = k$  and  $n$  be a non-negative integer. Then  $\text{range}(S) \leq n + 1$  if and only if  $\text{range}(Soc(M)/T) \leq n$ , for every subsocle  $T$  of  $M$  such that  $Soc(M) = T \oplus S$ .

**Proof:** Let  $\text{range}(S) \leq n + 1$ , then  $\text{Soc}(H_{k+n+1}(M)) \subseteq S \subseteq (H_k(M))$ . Trivially  $T \cap H_{k+n+1}(M) = 0$ . Hence, by Proposition 4.3.4,  $\text{Soc}(H_{k+n}(M/T)) \subseteq \text{Soc}(M)/T$ . It is trivial to see that  $\text{Soc}(M) = \text{Soc}(H_k(M)) + T$ , so by Proposition 4.3.4, we get  $\text{Soc}(M)/T \subseteq H_k(M/T)$ . Therefore,  $\text{range}(\text{Soc}(M)/T) \leq n$ .

Conversely, let  $\text{range}(\text{Soc}(M)/T) \leq n$ . Now we show that  $\text{Soc}(H_{k+n+1}(M)) \subseteq S$ . Let  $\text{Soc}(H_{k+n+1}(M)) \not\subseteq S$ , then by Proposition 4.3.5, we find a subsocle  $T$  such that  $\text{Soc}(M) = T \oplus S$  such that  $h(\text{Soc}(M)/T) = k$  and  $\text{Soc}(H_{k+n}(M/T)) \not\subseteq \text{Soc}(M)/T$  and hence  $\text{range}(\text{Soc}(M)/T) \not\leq n$ . Which is a contradiction.

Therefore,  $\text{Soc}(H_{k+n+1}(M)) \subseteq S$  and we get  $\text{range}(S) \leq n + 1$ .

## Section-4

### § 4.4. Center of $n$ - $h$ -purity

In this section we define a new concept of  $n$ - $h$ -purity which generalizes the concept of  $h$ -purity and obtain a characterization of center of  $n$ - $h$ -purity.

**Definition 4.4.1:** A submodule  $N$  of a QTAG-module  $M$  is called  $n$ - $h$ -pure in  $M$  if  $N/\text{Soc}^n(N)$  is  $h$ -pure in  $M/\text{Soc}^n(N)$ , where  $n$  is a non-negative integer.

It is evident that if  $n = 0$  then  $n$ - $h$ -purity is simply  $h$ -purity.

**Definition 4.4.2:** A subsocle  $S$  of a QTAG-module  $M$  is center of  $n$ - $h$ -purity if all complements of  $S$  in  $M$  are  $n$ - $h$ -pure submodules of  $M$ .

Firstly we prove the following:

**Theorem 4.4.3:** If  $N$  is a submodule of a QTAG-module  $M$ , then there is a complement of  $N$  which is  $h$ -pure in  $M$ .

**Proof:** It is sufficient to consider  $Soc(N) \neq Soc(M)$ . Suppose every uniform element of  $Soc(M)$  is of infinite height then trivially  $N \subseteq M^1$ . Now appealing to Theorem 1.4.8, we get a complement  $K$  of  $N$ , which is  $h$ -pure in  $M$ .

On the other hand if there is a uniform element  $x \in Soc(M)$  such that  $x \notin Soc(N)$  and  $H(x) < \infty$ . As if  $y \in Soc(M)$  such that  $y \notin Soc(N)$  and  $H(y) = \infty$ , then  $H(x + y) = H(x) < \infty$ .

Hence, appealing to Proposition 1.3.5, we shall get a summand  $K$  such that  $Soc(K) = (x + y)R$  and  $K \cap N = 0$ . Hence,  $K$  is  $h$ -pure in  $M$ .

**Theorem 4.4.4:** If  $S \subseteq Soc(M)$  then there exists a  $h$ -neat submodule  $K$  of  $M$  which is 1- $h$ -pure with  $Soc(K) = S$ .

**Proof:** Applying Theorem 4.4.3 for  $M/S$ , we get an  $h$ -pure submodule  $K/S$  in  $M/S$ , which is a complement of  $Soc(M)/S$ . Since  $(K/S) \cap (Soc(M)/S) = 0$ , for every uniform element  $x \in Soc(K)$ ,  $x + S = S$ , so  $x \in S$  and hence  $Soc(K) = S$ . Therefore,  $K$  is 1- $h$ -pure in  $M$ .

Now we show that  $K$  is  $h$ -neat. Let  $x$  be a uniform element in  $K \cap H_1(M)$ , then we get a uniform element  $y \in M$  such that  $d(yR/xR) = 1$ . Now if  $y \in K$  we get  $K$  to be  $h$ -neat submodule. Let  $y \notin K$  then  $((K + yR)/S) \cap (Soc(M)/S) \neq 0$  implies  $k + y + S = z + S$  for some  $z \in Soc(M)$ ,  $k \in K$ .

Hence,  $0 = H_1(zR) = H_1((k + y)R) = 0$ , so  $k + y \in Soc(M)$ . Therefore,  $H_1(kR) = H_1(yR) = xR$  and  $x \in H_1(K)$ . Hence,  $K$  is  $h$ -neat.

**Proposition 4.4.5:** Let  $S$  be a subsocle of a QTAG-module  $M$  such that  $S$  is center

of  $n$ - $h$ -purity for  $n \geq 1$ . Then  $Soc(M)/T$  is center of  $(n-1)$ - $h$ -purity in  $M/T$  for every complementary subsocle  $T$  of  $S$  in  $Soc(M)$ .

**Proof:** Let  $K/T$  be a complement of  $Soc(M)/T$  in  $M/T$ . Then trivially  $K \cap S = 0$ . Now we show that  $N \cap S \neq 0$  for  $K \subsetneq N$ .

Let  $N \cap S = 0$  then we show that  $N/T \cap (S \oplus T)/T = 0$ . Let on contrary  $N/T \cap (S \oplus T)/T \neq 0$ , then  $x + T = s + T$  where  $x \in N$ ,  $s \in S$  and we get  $x - s \in T \subseteq K \subseteq N$ , consequently,  $s \in N \cap S = 0$  and  $x + T = T$ . Which is a contradiction. Therefore,  $K$  is a complement of  $S$ . Hence  $K/Soc^n(K)$  is  $h$ -pure in  $M/Soc^n(K)$ . Now we show that  $(K/T)/(Soc^{(n-1)}(K/T))$  is  $h$ -pure in  $(M/T)/(Soc^{(n-1)}(K/T))$ . It is easy to see that  $Soc(K) = T$  and  $Soc^{(n-1)}(K/T) \subseteq Soc^n(K)/T$ . Now for any uniform element  $x \in Soc^n(K)$ , let  $yR = Soc(xR)$  then  $H_{n-1}(xR) = yR$ . Hence

$$\begin{aligned} H_{n-1}(\bar{x}R) &= H_{n-1}((xR + T)/T) \\ &= (H_{n-1}(xR) + T)/T \\ &= \bar{0} \end{aligned}$$

Therefore,  $Soc^{(n-1)}(K/T) = Soc^n(K)/T$ . Further, under the canonical isomorphism

$$\begin{aligned} (M/T)/(Soc^{(n-1)}(K/T)) &= (M/T)/(Soc^n(K)/T) \\ &\cong M/Soc^n(K) \end{aligned}$$

Therefore,  $(K/T)/(Soc^{(n-1)}(K/T))$  is mapped onto  $K/Soc^n(K)$ . Hence,  $K/T$  is  $(n-1)$ - $h$ -pure in  $M/T$  and we get the result.

Now we prove the main result of this section:

**Theorem 4.4.6:** A subsocle  $S$  of a QTAG-module  $M$  is center of  $n$ - $h$ -purity for some  $n \geq 0$  if and only if either  $h(S) = \infty$ , or  $S$  is open subsocle of  $M$  such that  $\text{range}(S) \leq n + 2$ .

**Proof:** Let  $S$  be a center of  $n$ - $h$ -purity and  $h(S) < \infty$ . Suppose  $h(S) = k$ , then we show that  $Soc(H_{k+n+2}(M)) \subseteq S$ , which in turn will imply  $range(S) \leq n + 2$ . Let  $Soc(H_{k+n+2}(M)) \not\subseteq S$ , then appealing to Proposition 4.3.5, we will find a subsocle  $T$  such that  $Soc(M) = S \oplus T$ ,  $h(Soc(M)/T) = k$  and  $Soc(H_{k+n+1}(M/T)) \not\subseteq Soc(M)/T$ . As remarked in section 3, for  $n = 0$ ,  $range(S) \leq 2$ , so we use induction. However, appealing to Proposition 4.4.5, we get  $Soc(M)/T$  as center of  $(n - 1)$ - $h$ -purity. Therefore,  $range(Soc(M)/T) \leq n - 1 + 2 = n + 1$ , consequently,

$$Soc(H_{k+n+1}(M/T)) \subseteq Soc(M)/T$$

Which is a contradiction. Hence,  $range(S) \leq n + 2$ .

Conversely, if  $h(S) = \infty$  then by Theorem 1.4.8,  $S$  is center of  $h$ -purity and hence for  $n = 0$ ,  $S$  is center of  $n$ - $h$ -purity. Suppose  $range(S) \leq n + 2$  and  $Soc(H_{k+n+2}(M)) \subseteq S \subseteq H_k(M)$ . Let  $K$  be a complement of  $S$  in  $M$ . Now we prove that

$$\begin{aligned} Soc(H_{k+2}(M/Soc^n(K))) &\subseteq (Soc(M) + Soc^n(K))/Soc^n(K) \\ &\subseteq H_k(M/Soc^n(K)) \end{aligned}$$

For any uniform element  $x \in H_{k+2}(M)$ , let  $\bar{x} \in Soc(H_{k+2}(M/Soc^n(K)))$ . Then  $H_1(\bar{x}R) = 0$ , hence  $H_1(xR) \subseteq K$ , but due to Proposition 1.3.21,  $K$  is  $h$ -neat and so there is a uniform element  $t \in K$  such that  $H_1(xR) = H_1(tR) = zR$ . Now as  $x \in H_{k+2}(M)$ , there is a uniform element  $y \in M$  such that  $d(yR/xR) = k + 2$ , consequently,  $H_{k+3}(yR) = H_1(tR) = zR$  and we get

$$\begin{aligned} H_{k+3+n-1}(yR) &= H_n(tR) \\ &= H_{n-1}(zR) \end{aligned}$$

but

$$\begin{aligned} H_{k+n+2}(yR) &= H_n(tR) \\ &\subseteq K \cap H_{k+n+2}(M) \\ &= 0 \end{aligned}$$

Hence,  $t \in Soc^n(K)$ . Further, as  $H_1(xR) = H_1(tR)$ , we get  $x - t \in Soc(M)$ .

Therefore,  $x - t + Soc^n(K) = x + Soc^n(K) = \bar{x} \in (Soc(M) + Soc^n(K))/Soc^n(K)$  and we get the first inclusion. Trivially,

$$H_k(M/Soc^n(K)) = (H_k(M) + Soc^n(K))/Soc^n(K)$$

and as  $K$  is complement of  $S$ ,  $Soc(M) = S + Soc(K)$ . Therefore, the second inclusion also follows. Hence,

$$\text{range}\left((Soc(M) + Soc^n(K))/Soc^n(K)\right) \leq 2$$

and we get  $(Soc(M) + Soc^n(K))/Soc^n(K)$  as center of  $h$ -purity in  $M/Soc^n(K)$ . Further, it is easy to see that  $K/Soc^n(K)$  is complement of  $(Soc(M) + Soc^n(K))/Soc^n(K)$  in  $M/Soc^n(K)$  and hence  $K/Soc^n(K)$  is  $h$ -pure submodule of  $M/Soc^n(K)$ .

Therefore,  $S$  is center of  $n$ - $h$ -purity.

## Section-5

### § 4.5. Special QTAG-Module

In this section we study the implications of the consequence of imposition of some restrictions on heights of the elements of QTAG-module and some characterizations in this regard has been obtained.

First of all we defining the following:

**Definition 4.5.1:** A module  $M$  is said to be *special* if and only if

$$H(x + y) \leq H(x) + H(y),$$

for each uniform element  $x, y \in M$ . Such that  $x + y \neq 0$ .

To start with we need the following lemmas.

**Lemma 4.5.2:** If  $M$  is special, then every uniform element of zero height is of exponent 1.

**Proof:** Let  $y \in M$  such that  $H(y) = 0$ . Suppose  $e(y) \neq 1$ , therefore there exists  $x \in yR$  such that  $d(yR/xR) = 1$ . Let  $x = yr$ , for  $r \in R$ , then

$$\begin{aligned} 1 &\leq H(x) \\ &= H(yr) \\ &= H(yr - y + y) \\ &= H(y(r - 1) + y) \\ &\leq H(y(r - 1)) + H(y) \end{aligned}$$

Since  $yR$  is totally ordered, either  $y(r - 1)R = yR$  or  $y(r - 1)R = xR$ . But  $y(r - 1)R \neq xR$  yields  $y(r - 1)R = yR$ . Consequently,

$$1 \leq H(x) \leq H(y(r - 1)) + H(y) = 2H(y) = 0.$$

Which shows a contradiction. Hence,  $e(y) = 1$ . Thus every element of zero height is of exponent 1.

**Lemma 4.5.3:** If  $M$  is special, then every uniform element of finite height has zero height.

**Proof:** Let  $x \in M$  and  $H(x) = n < \infty$ . Then there exists  $y \in M$  such that  $x \in yR$  and  $d(yR/xR) = n$  with  $H(y) = 0$ . As, from Lemma 4.5.2,  $y$  is of exponent 1, we have  $xR = yR$ . Hence,  $H(x) = 0$ . Thus in a special QTAG-module every element of finite height has zero height.



**Lemma 4.5.4:** If  $M$  is  $h$ -reduced and special, then every uniform element has zero height and is of exponent 1.

**Proof:** Let  $M^1$  be a submodule of  $M$  generated by all the uniform elements of infinite height.

Now we shall prove that  $M^1$  is  $h$ -pure. Suppose on contrary that  $M^1$  is not  $h$ -pure, then there exists  $0 \neq x \in M^1$  such that  $H_{M^1}(x) \neq H_M(x)$ , therefore  $x$  has finite height in  $M^1$ . Since  $x \in M^1$ , therefore  $x$  has infinite height in  $M$ , then there exists  $y \in M$ ,  $y \notin M^1$ , such that  $d(yR/xR) = n + 1$  and consequently,  $x = yr$  for some  $r \in R$ . Since  $y \notin M^1$ , it has finite height and as we did in Lemma 4.5.2,

$$\begin{aligned}
\infty &= H(x) \\
&= H(yr) \\
&= H(yr - y + y) \\
&= H(y(r - 1) + y) \\
&\leq H(y(r - 1)) + H(y) \\
&= 2H(y) \\
&\leq \infty
\end{aligned}$$

Which is a contradiction. Therefore,  $M^1$  is  $h$ -pure.

Hence, height of every element of  $M^1$  in  $M^1$  is same as in  $M$ . Thus  $M^1$  is  $h$ -divisible. Since  $M$  is  $h$ -reduced therefore  $M^1 = (0)$ . Thus every non-zero element of  $M$  has finite height. The result follows from Lemma 4.5.2 and Lemma 4.5.3.

Now we are able to prove the main result.

**Theorem 4.5.5:** If  $M$  is special, then it is either  $h$ -divisible or  $h$ -reduced.

**Proof:** Let  $T$  be the maximal  $h$ -divisible submodule of  $M$ , then by Theorem 1.4.6,

we have  $M = T \oplus S$ , where  $S$  is  $h$ -reduced. If  $S$  has no element of finite height in  $M$ , then  $M$  has no element of finite height. Therefore, either  $M$  has all elements of infinite height or  $S$  has an element of finite height, hence,  $M$  is  $h$ -divisible.

Now we consider the case where  $M$  is not  $h$ -divisible. Suppose  $T$  and  $S$  both are nontrivial. For any  $t \in T$ ,  $H(t) = \infty$ ; also there exists  $s \in S$  such that  $H(s) < \infty$ . Applying Lemma 4.5.2 and Lemma 4.5.3, we have  $H(s) = 0$  and exponent of  $s$  is 1. Let us consider  $P = \text{ann}(sR)$ , then  $sP = 0$ . Now choosing  $r \in P$ , we have  $t = (s(r - 1) + s + t)$ , and consequently,

$$\begin{aligned} \infty &= H(t) \\ &= H(s(r - 1) + s + t) \\ &\leq H(s(r - 1)) + H(s + t) \end{aligned}$$

Now suppose that  $H(s + t) = n$  i.e.,  $s + t \in H_n(M)$  implying thereby  $s \in H_n(M)$  as  $t \notin H_n(M)$ . Thus,  $H(s) \geq n$ . But  $H(s) = 0$  yields  $n = 0$ . Hence,

$$\infty = H(t) \leq 0 + 0 = 0.$$

Which shows a contradiction.

Therefore, the supposition that both  $T$  and  $S$  are nontrivial is false, and since  $M$  is not  $h$ -divisible, so is  $h$ -reduced.

**Proposition 4.5.6:** If every uniform element of  $M$  is of exponent 1, then  $M$  is decomposable.

**Proof:** Let  $F$  be the set of all linearly independent subsets of  $M$ , then trivially  $F$  will contain a maximal element  $B$  of  $F$ .

Let  $C(B) = \{T : T \text{ is a cyclic module generated by } x \in B\}$ . Let  $M' = \sum T$ , where  $T \in C(B)$ . Then trivially  $M' = \oplus \sum T$ . Now suppose that  $x \in M$ ,  $x \notin M'$ , then there exists  $r \in R$  such that  $xr \notin M'$ , where  $xrR = xR$ , as  $x$  is of exponent 1.

Consider  $B \cup xR$ ,  $B$  is a proper subset of  $B \cup xR$ . If for some

$$r_1, r_2, \dots, r_k \in R, \quad x_1 r_1 + x_2 r_2 + \dots + x_k r_k + xr = 0,$$

where  $x_i r_i R = x_i R$ ,  $xrR = xR$  and  $x_i \in B$ , then  $xr = 0$ . Otherwise,  $xr \in B$ , which shows a contradiction.

Hence,  $xr = 0$  and  $x_i r_i = 0$ , for all  $i$ . Thus  $B \cup xR$  is linearly independent, which contradicts the maximality of  $B$ , hence  $x \in M'$ . So we get  $M = M' = \sum T$ , where  $T$  is a cyclic module and  $M = \oplus \sum T$ . Hence  $M$  is decomposable.

**Corollary 4.5.7:** If  $M$  is special, then it is either h-divisible or decomposable.

**Proof:** The result follows from Lemma 4.5.3, Lemma 4.5.4, Theorem 4.5.5 and Proposition 4.5.6.

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